2.1. 확률변수 X 와 Y 의 결합확률밀도함수가

$$f_{1,2}(x,y) = 10xy^2 I_{(0<x<y<1)}$$

일 때 다음 확률을 구하여라.

(a) $P(Y \le 1/2)$

$$f(y) = \int_0^y 10xy^2\,dx = \left[5x^2y^2\right]_0^y = 5y^4 I_{(0<y<1)}$$

$$P(Y \le 1/2) = \int_0^{1/2} f(y)\,dy = \int_0^{1/2} 5y^4\,dy = \left[y^5\right]_0^{1/2} = \frac{1}{32}$$

(b) $P(X+Y \le 1) = P(X \le 1-Y)$

$$= \int_0^1 \int_0^{1-y} 10xy^2\,dx\,dy = \int_0^1 \left[5x^2y^2\right]_0^{1-y} dy$$

$$= \int_0^1 5(1-y)^2y^2\,dy$$

$$= \left[5y^2 \cdot -\tfrac{1}{3}(1-y)^3\right]_0^1 - \int_0^1 10y \cdot -\tfrac{1}{3}(1-y)^3\,dy$$

$$= \frac{10}{3}\int_0^1 y(1-y)^3\,dy$$

$$= 10/3\left\{\left[y \cdot -\tfrac{1}{4}(1-y)^4\right]_0^1 - \int_0^1 -\tfrac{1}{4}(1-y)^4\,dy\right\}$$

$$= \frac{10}{3}\left[-\frac{1}{20}(1-y)^5\right]_0^1 = \frac{1}{6}$$ △ (half correct)

Conside the range of x: $0<x<y$

[Textbook Solution]

$$= \int_0^1 \int_0^{y\wedge(1-y)} 10xy^2\,dx\,dy = \int_0^1 \left[5x^2y^2\right]_0^{y\wedge(1-y)} dy$$

$$= \int_0^1 5(y\wedge(1-y))^2y^2\,dy$$

$$= \int_0^{1/2} 5y^4\,dy + \int_{1/2}^1 5(1-y)^2y^2\,dy$$

2.2. 확률변수 X 와 Y 의 결합확률밀도함수가

$$f_{1,2}(x,y) = Cx^2y\, I_{(0<x<y<1)}$$

일 때 다음의 값들을 구하여라

(a) 상수 C : $\int_0^1\int_0^y Cx^2y\,dx\,dy = \int_0^1 \left[\frac{C}{3}x^3y\right]_0^y dy = \int_0^1 C\cdot\frac{1}{3}\cdot y^4\,dy$

$$= \left[C\cdot\tfrac{1}{3}\cdot\tfrac{1}{5}y^5\right]_0^1 = C\cdot\frac{1}{15} = 1$$

$$\therefore C = 15$$

(b) $P(Y \le 1/2) = \int_0^{1/2}\int_0^y 15x^2y\,dx\,dy = \int_0^{1/2}\left[5x^3y\right]_0^y dy = \int_0^{1/2} 5y^4\,dy$

$$= \left[y^5\right]_0^{1/2} = \frac{1}{32}$$

(c) $P(Y \le 2X) = P(X < Y \le 2X \wedge 1)$

$$= \int_0^1 \int_x^{2x\wedge1} 15x^2y\,dy\,dx = 15\int_0^1 x^2\int_x^{2x\wedge1} y\,dy\,dx$$

$$= 15\int_0^1 x^2\left[\tfrac{1}{2}y^2\right]_x^{2x\wedge1} dy\,dx$$

$$= 15\int_0^1 x^2\cdot\frac{1}{2}\cdot\left[(2x\wedge1)^2 - x^2\right]dx$$

$$= \frac{15}{2}\left[\int_0^{1/2} x^2\left((2x)^2 - x^2\right)dx + \int_{1/2}^1 x^2(1-x^2)\,dx\right]$$

$$= \frac{15}{2}\left[\int_0^{1/2} 3x^4\,dx + \int_{1/2}^1 x^2 - x^4\,dx\right]$$

$$= \frac{15}{2}\left[\left[\tfrac{3}{5}x^5\right]_0^{1/2} + \left[\tfrac{1}{3}x^3 - \tfrac{1}{5}x^5\right]_{1/2}^1\right]$$

$$= \frac{15}{2}\left[\tfrac{3}{5}\cdot\tfrac{1}{32} + \tfrac{1}{3} - \tfrac{1}{5} - \tfrac{1}{3}\cdot\tfrac{1}{8} + \tfrac{1}{5}\cdot\tfrac{1}{32}\right]$$

$$= \frac{9}{64} + \frac{5}{2} - \frac{3}{2} - \frac{5}{16} + \frac{3}{64}$$

$$= \frac{12}{64} + \frac{11}{16} = \frac{12+44}{64} = \frac{56}{64} = \frac{7}{8}$$ ✓✓

[Textbook Solution]

$$P(Y \le 2X) = P(Y/2 \le X < Y) = \int_0^1 \int_{y/2}^y 15x^2y\,dx\,dy$$

$$= \int_0^1 \left[5x^3y\right]_{y/2}^y dy = \int_0^1 5y^4 - \frac{5}{8}y^4\,dy$$

$$= \int_0^1 \frac{35}{8}y^4\,dy = \left[\frac{7}{8}y^5\right]_0^1 = \frac{7}{8}$$

2.3. 확률변수 X 와 Y 의 결합확률밀도함수가

$$f_{1,2}(x,y) = e^{-y} I_{(0 \le x \le y \le \infty)}$$

일 때 다음 확률을 구하여라.

(a) $P(X \le 1)$

$$f_1(x) = \int_x^\infty e^{-y}\,dy = e^{-x}$$

$$P(X \le 1) = \int_0^1 e^{-x}\,dx = 1 - e^{-1}$$

(Took me 1 hour)

☆ (b) $P(X+Y \le 1) = P(X \le 1-Y \wedge Y)$ $[= P(Y \le 1-X)]$

#1 $= P(X \le Y \le 1-X) = \int_0^{1/2}\int_x^{1-x} e^{-y}\,dy\,dx$

$$= \int_0^{1/2}\left[-e^{-y}\right]_x^{1-x} dx \qquad = \left[-e^{-x} - e^{-(1-x)}\right]_0^{1/2}$$

$$= \int_0^{1/2} e^{-x} - e^{-(1-x)}\,dx \qquad = -e^{-1/2} - e^{-1/2} + 1 + e^{-1}$$

$$= 1 - 2e^{-1/2} + e^{-1}$$

✱ $= \left[-e^{-x} + \frac{1}{1-x}e^{-(1-x)}\right]_0^{1/2}$ incorrect

$$= -e^{-1/2} + 2e^{-1/2} + 1 - e^{-1}$$

$$= 1 + e^{-1/2} - e^{-1}$$

#2 $P(X+Y\le 1) = P(X \le 1-Y)$

$= P(0 \le X \le 1-Y \wedge Y)$

$= \int_0^1 \int_0^{(1-y)\wedge y} e^{-y}\,dx\,dy = \int_0^1 \left[e^{-y}x\right]_0^{(1-y)\wedge y} dy$

$= \int_0^1 e^{-y}(1-y)\wedge y\,dy = \int_0^{1/2} ye^{-y}dy + \int_{1/2}^1 (1-y)e^{-y}dy$

$= \left[-ye^{-y}\right]_0^{1/2} - \int_0^{1/2} -e^{-y}dy + \left[-e^{-y}\right]_{1/2}^1 + \int_{1/2}^1 -ye^{-y}dy$

$= -\frac{1}{2}e^{-1/2} + \left[-e^{-y}\right]_0^{1/2} - e^{-1} + e^{-1/2} - \int_{1/2}^1 ye^{-y}dy$

$= -\frac{1}{2}e^{-1/2} - e^{-1/2} + 1 - e^{-1} + e^{-1/2} - \left\{\left[-ye^{-y}\right]_{1/2}^1 - \int_{1/2}^1 -e^{-y}dy\right\}$

$= -\frac{1}{2}e^{-1/2} - e^{-1} + 1 - \left\{-e^{-1} + \frac{1}{2}e^{-1/2} + \left[-e^{-y}\right]_{1/2}^1\right\}$

$= -\frac{1}{2}e^{-1/2} - e^{-1} + 1 + e^{-1} - \frac{1}{2}e^{-1/2} + e^{-1} - e^{-1/2}$

$= 1 - 2e^{-1/2} + e^{-1}$ ✓✓

* Key takeaway: deciding which r.v. to integrate first reduces/increases calc. time.

2.4. 확률변수 X와 Y의 결합확률밀도함수가 다음과 같을 때, X의 주변확률밀도함수를 구하여라

(a) $f_{1,2}(x,y) = 2\,I_{(0<x<y<1)}$

$f_1(x) = \int_x^1 2\,dy = \left[2y\right]_x^1 = 2(1-x)\,I_{(0<x<1)}$ ✓✓

(b) $f_{1,2}(x,y) = e^{-y}I_{(0<x<y<\infty)}$

$f_1(x) = \int_x^\infty e^{-y}dy = \left[-e^{-y}\right]_x^\infty = e^{-x}I_{(0<x<\infty)}$ ✓✓

2.5 확률변수 X와 Y의 결합누적분포가 다음과 같을 때, X와 Y의 결합확률밀도함수를 구하여라

(a) $F_{1,2}(x,y) = \begin{cases} x^2(1-e^{-2y}) & 0\le x\le 1,\ y\ge 0 \\ (1-e^{-2y}) & x\ge 1,\ y\ge 0 \\ 0 & y<0 \end{cases}$

$F_{1,2}(x,y) = P(0\le X\le x,\ 0\le Y\le y)$

$\int_0^y\int_0^x f_{1,2}(x,y)\,dx\,dy = F_{1,2}(x,y)$

$\therefore \frac{\partial}{\partial y}\frac{\partial}{\partial x}F_{1,2}(x,y) = f_{1,2}(x,y)$

$f_{1,2}(x,y) = \begin{cases} 4xe^{-2y} & 0\le x\le 1,\ y\ge 0 \\ 0 & x\ge 1,\ y\ge 0 \text{ (or) } y<0 \end{cases}$ ✓✓

$\int_0^y\int_0^x f_{1,2}(x,y)\,dx\,dy = \int_0^y \left[2x^2e^{-2y}\right]_0^x dy$

$= \int_0^y 2x^2e^{-2y}dy$

$= \left[-x^2e^{-2y}\right]_0^y = x^2(1-e^{-2y})$

(b) $F_{1,2}(x,y) = \begin{cases} 1-2e^{-y}-e^{-2x}+2e^{-x-y} & 0<x<y \\ 1-2e^{-y}+e^{-2y} & 0<y\le x \\ 0 & x\le 0 \text{ or } y\le 0 \end{cases}$

$F_{1,2}(x,y) = P(X\le x, Y\le y)$

$(0<x<y) := \int_0^y\int_0^{x\wedge y} f_{1,2}(x,y)\,dx\,dy$

$\frac{\partial}{\partial y}F_{1,2}(x,y) = \frac{\partial}{\partial y}(1-2e^{-y}-e^{-2x}+2e^{-x}e^{-y})$

$= 2e^{-y} - 2e^{-x}e^{-y}$

$= 2e^{-y}(1-e^{-x})$

$\frac{\partial}{\partial x}\frac{\partial}{\partial y}F_{1,2}(x,y) = 2e^{-y}e^{-x} = 2e^{-(x+y)}I_{(0<x<y)}$ ✓✓

$(0<y<x)$: $\int_0^x\int_0^{y\wedge x} f_{1,2}(x,y)\,dy\,dx$

$\frac{\partial}{\partial x}F_{1,2}(x,y) = 1$, $\frac{\partial}{\partial y}\frac{\partial}{\partial x}F_{1,2}(x,y) = 0$

(double check)

Suppose $f_{1,2}(x,y) = 2e^{-(x+y)}I_{(0<x<y)}$

$F_{1,2}(x,y) = \int_0^y\int_0^{x\wedge y} 2e^{-(x+y)}\,dx\,dy$

if $x>y$, $F_{1,2}(x,y) = \int_0^y\int_0^y 2e^{-x}e^{-y}\,dx\,dy$

$= 2\int_0^y e^{-y}\left[-e^{-x}\right]_0^y dy$

$= 2\int_0^y e^{-y}(1-e^{-y})dy = 2\int_0^y e^{-y}-e^{-2y}dy = 2\left[-e^{-y}+\frac{1}{2}e^{-2y}\right]_0^y$

$= 2\left\{-e^{-y}+\frac{1}{2}e^{-2y}+1-\frac{1}{2}\right\}$

$= 1-2e^{-y}+e^{-2y}$

2.6. 확률변수 X와 Y의 결합확률밀도함수가

$f_{1,2}(x,y) = xe^{-y}I_{(0<x<y<\infty)}$

일 때 X와 Y의 공분산 $Cov(X,Y)$를 구하여라.

$Cov(X,Y) = E[XY] - E[X]E[Y]$

$E[XY] = \int_0^\infty\int_0^y xy\cdot xe^{-y}\,dx\,dy = \int_0^\infty ye^{-y}\int_0^y x^2\,dx\,dy$

$= \int_0^\infty ye^{-y}\left[\frac{1}{3}x^3\right]_0^y dy = \int_0^\infty ye^{-y}\cdot\frac{1}{3}y^3dy$

$= \int_0^\infty \frac{1}{3}y^4e^{-y}dy = \frac{1}{3}\int_0^\infty y^4e^{-y}dy$

$\alpha! = \int_0^\infty x^\alpha e^{-x}dx$ $\therefore \frac{1}{3}\int_0^\infty y^4e^{-y}dy = \frac{4!}{3} = 4\cdot 2 = 8$

$E[X] = \int_0^\infty\int_0^y x^2e^{-y}\,dx\,dy = \int_0^\infty e^{-y}\left[\frac{1}{3}x^3\right]_0^y dy = \int_0^\infty \frac{1}{3}y^3e^{-y}dy = 2$

$E[Y] = \int_0^\infty\int_0^y xye^{-y}dx\,dy = \int_0^\infty ye^{-y}\left[\frac{1}{2}x^2\right]_0^y dy = \int_0^\infty \frac{1}{2}y^3e^{-y}dy = 3$

$\therefore Cov(X,Y) = E[XY] - E[X]E[Y] = 8 - 2\cdot 3 = 2$ ✓✓

2.7. 확률변수 X 와 Y의 결합확률밀도함수가

$$f_{1,2}(x,y) = e^{-y} I_{(0<x<y<\infty)}$$

일 때 다음을 구하여라.

(a) X 와 Y의 결합적률생성함수 $M(t_1,t_2)$

(b) X 와 Y의 상관계수 $Corr(X,Y)$

(a) $M(t_1,t_2) = E[e^{t_1X+t_2Y}]$

$$= \int_0^\infty \int_0^y e^{t_1x+t_2y} e^{-y}\, dx\, dy = \int_0^\infty e^{-(1-t_2)y} \int_0^y e^{t_1x}\, dx\, dy$$

$$= \int_0^\infty e^{-(1-t_2)y} \left[\frac{1}{t_1} e^{t_1x}\right]_0^y dy$$

$$= \int_0^\infty e^{-(1-t_2)y}\, t_1^{-1}\left(e^{t_1y} - 1\right) dy$$

$$= t_1^{-1}\int_0^\infty -e^{-(1-t_2)y} + e^{-(1-t_1-t_2)y}\, dy$$

$$= t_1^{-1}\left[-(1-t_2)^{-1} e^{-(1-t_2)y} + (1-t_1-t_2) e^{-(1-t_1-t_2)y}\right]_0^\infty$$

$$= t_1^{-1}\left[(1-t_2)^{-1} + (1-t_1-t_2)^{-1}\right]$$

$$= -\frac{1}{t_1}\left(\frac{(1-t_1-t_2)-(1-t_2)}{(1-t_2)(1-t_1-t_2)}\right)$$

$$= (1-t_2)^{-1}(1-t_1-t_2)^{-1} \qquad t_2<1,\ t_1+t_2<1$$ ✓✓

$$C_{X,Y}(t_1,t_2) = -\log(1-t_2) - \log(1-t_1-t_2)$$

$* -\log(1+A) = A + \frac{A^2}{2} + \frac{A^3}{3} + \cdots \quad (|A|<1)$

$$= \left(t_2 + \frac{t_2^2}{2} + \frac{t_2^3}{3} + \cdots\right) + \left| t_1+t_2 + (t_1+t_2)^2\frac{1}{2} + (t_1+t_2)^3\frac{1}{3} + \cdots\right)$$

$$= t_1 + 2t_2 + t_2^2 + t_1t_2 + t_2^2\cdot\frac{1}{2} + \cdots \qquad |t_2|<1,\ |t_1+t_2|<1$$

$$C_{X,Y}(t_1,t_2) = \sum_r^\infty \sum_s^\infty \frac{c_{r,s}}{r!s!} t_1^r t_2^s$$

$c_{1,0}=1 \quad c_{0,1}=2$

$c_{0,2}=2 \quad c_{2,0}=1 \quad c_{1,1}=1$

$Cov(X,Y)=1 \quad Var(X)=1 \quad Var(Y)=2$

$Corr(X,Y) = Cov(X,Y)/\sqrt{Var(X)Var(Y)} = 1/\sqrt{2}$ ✓✓

2.8. 확률변수 X 와 Y의 결합확률밀도함수가 $f_{1,2}(x,y) = 10xy^2 I_{(0<x<y<1)}$ 일 때 다음을 구하여라.

(a) Y의 주변확률밀도함수

(b) $Y=y\ (0<y<1)$ 가 주어진 조건에서 X의 조건부확률밀도함수 $f_{1|2}(x|y)$

(c) $E[X|Y]$, $Var[X|Y]$

(d) $Var[E[X|Y]]$, $E[Var(X|Y)]$

(e) X의 주변확률밀도함수 $f_1(x)$ 와 $Var(X)$

$$f_{1,2}(x,y) = 10xy^2 I_{(0<x<y<1)}$$

(a) $f_2(y) = \int_0^y 10xy^2\, dx = \left[5x^2y^2\right]_0^y = 5y^4 I_{(0<y<1)}$ ✓✓

(b) $f_{1|2}(x|y) = f_{1,2}(x,y)/f_2(y) = 2xy^{-2} I_{(0<x<y)}$ ✓✓

(c) $E[X|Y] = \int_0^y x\cdot 2xy^{-2}\, dx = \int_0^y 2x^2y^{-2}\, dx = \left[\frac{2}{3}x^3y^{-2}\right]_0^y = \frac{2}{3}y$

$E[X^2|Y] = \int_0^y x^2\cdot 2xy^{-2}\, dx = \int_0^y 2x^3y^{-2}\, dx = \left[\frac{1}{2}x^4y^{-2}\right]_0^y = \frac{1}{2}y^2$

$Var(X|Y) = E[X^2|Y] - E[X|Y]^2 = \frac{1}{2}y^2 - \frac{4}{9}y^2 = \frac{1}{18}y^2$ ✓✓

(d) $Var(E[X|Y]) = Var(\frac{2}{3}y) = \frac{4}{9}Var(y)$

$E[Var(X|Y)] = E[\frac{1}{18}y^2] = \frac{1}{18}E[y^2]$

$E[y] = \int_0^1 y\cdot 5y^4\, dy = \left[\frac{5}{6}y^6\right]_0^1 = 5/6$ ✓✓

$E[y^2] = \int_0^1 y^2\cdot 5y^4 dy = \left[\frac{5}{7}y^7\right]_0^1 = 5/7$ ✓✓

$Var(Y) = E[Y^2] - E[Y]^2 = 5/7 - 25/36 = \frac{180-175}{7\cdot 36} = \frac{5}{252}$

$\therefore \frac{4}{9}Var(Y) = \frac{1}{9}\cdot\frac{5}{252} = \frac{5}{9\cdot 63}$

$\frac{1}{18}E[y^2] = \frac{1}{18}\cdot\frac{5}{7} = \frac{5}{70+56} = \frac{5}{126}$

(e) $\therefore Var(X) = Var(E[X|Y]) + E[Var(X|Y)] = \frac{5}{9\cdot 63} + \frac{5}{126} = \frac{55}{1134}$ ✓✓

$f_1(x) = \int_x^1 10xy^2\, dy = \left[\frac{10}{3}xy^3\right]_x^1 = \frac{10}{3}x(1-x^3) I_{(0<x<1)}$

$= \frac{10}{3}x - \frac{10}{3}x^4 I_{(0<x<1)}$ ✓✓

$E[X] = \int_0^1 \frac{10}{3}x^2 - \frac{10}{3}x^5\, dx = \left[\frac{10}{9}x^3 - \frac{10}{18}x^6\right]_0^1 = \frac{10}{18}$

$E[X^2] = \int_0^1 \frac{10}{3}x^3 - \frac{10}{3}x^6\, dx = \left[\frac{10}{12}x^4 - \frac{10}{21}x^7\right]_0^1 = \frac{10}{12} - \frac{10}{21}$

$Var(X) = \frac{10}{12} - \frac{10}{21} - \left(\frac{10}{18}\right)^2 = \frac{55}{1134}$ ✓✓

2.9. 확률변수 X와 Y의 결합확률밀도함수가 $f_{1,2}(x,y) = e^{-y} I_{(0<x<y<\infty)}$ 일 때 다음을 구하여라

(a) Y의 주변확률밀도함수 $f_2(y)$

(b) $Y=y\ (y>0)$ 가 주어진 조건에서 X의 조건부확률밀도함수 $f_{1|2}(x|y)$

(c) $E[X|Y]$, $Var(X|Y)$

(d) $Var[E[X|Y]]$, $E[Var(X|Y)]$

(e) X의 주변확률밀도함수 $f_1(x)$ 와 $Var(X)$

(a) $f_2(y) = \int_0^y e^{-y}\,dx = e^{-y}\,y\,I_{(0,y)}(x)$

(b) $f_{1|2}(x|y) = f_{1,2}(x,y) / f_2(y) = y^{-1} I_{(0<x<y<\infty)}$

(c) $E[X|Y] = \int_0^y x\, f_{1|2}(x|y)\,dx = \left[\frac{1}{2}x^2\right]_0^y y^{-1} I_{(0<x<y<\infty)} = \frac{1}{2}y$ ✓✓

$Var(X|Y) = \frac{1}{12}Y^2$

(d) $Var(E[X|Y])$, $E[Var(X|Y)]$

$Var(E[X|Y]) = Var(\frac{1}{2}y) = \frac{1}{4}Var(y)$

$E[Var(X|Y)] = E[E[X^2|Y] - (E[X|Y])^2]$
$= E[\frac{1}{3}y^2 - \frac{1}{4}y^2]$
$= \frac{1}{12}E[y^2]$

$E[y] = \int_0^\infty y\cdot f_2(y)\,dy = \int_0^\infty y^3 e^{-y}\,dy = 2! = 2$
$E[y^2] = \int_0^\infty y^2 \cdot f_2(y)\,dy = \int_0^\infty y^3 e^{-y}\,dy = 3! = 6$
$Var(y) = E[y^2] - E[y]^2 = 6 - 2^2 = 2$

$\therefore Var(E[X|Y]) = \frac{1}{4}Var(y) = \frac{1}{2}$ ✓
$E[Var(X|Y)] = \frac{1}{12}E[y^2] = \frac{6}{12} = \frac{1}{2}$ ✓

$Var(X) = Var(E[X|Y]) + E[Var(X|Y)] = \frac{1}{2} + \frac{1}{2} = 1$ ✓✓

(e) $f_1(x) = \int_x^\infty e^{-y}\,dy = e^{-x} I_{(0<x<\infty)}$

$E[X] = 1$, $E[X^2] = 2! = 2$ $\therefore Var(X) = 1$

2.10. 확률변수 X와 Y의 결합확률밀도함수가 다음과 같은 경우에 $Var(E[X|Y])$ 와 $E[Var(X|Y)]$를 구하여라.

(a) $f_{1,2}(x,y) = 2 I_{(0<x<y<1)}$

$f_Y(y) = \int_0^y 2\,dx = [2x]_0^y = 2y\, I_{(0<y<1)}$

$f_{1|2}(x|y) = f_{1,2}(x,y) / f_2(y) = 2/2y = y^{-1} I_{(0,y)}(x)$

$E[X|Y] = \int_0^y x\cdot f_{1|2}(x,y)\,dx = \int_0^y x y^{-1}\,dx = \left[\frac{1}{2}x^2 y^{-1}\right]_0^y = \frac{1}{2}y\, I_{(0<y<1)}$

$E[X^2|Y] = \int_0^y x^2 \cdot y^{-1}\,dx = \left[\frac{1}{3}x^3 y^{-1}\right]_0^y = \frac{1}{3}y^2$

$Var(X|Y) = E[X^2|Y] - E[X|Y]^2 = \frac{1}{3}y^2 - (\frac{1}{2}y)^2 = \frac{1}{12}y^2$

$Var(E[X|Y]) = \frac{1}{4}Var(y) = \frac{1}{4}\cdot\frac{1}{18} = \frac{1}{72}$ ✓✓
$E[Var(X|Y)] = E[\frac{1}{12}y^2] = \frac{1}{12}E[y^2] = \frac{1}{12}\cdot\frac{1}{2} = \frac{1}{24}$ ✓✓

$E[y] = \int_0^1 2y^2\,dy = [\frac{2}{3}y^3]_0^1 = \frac{2}{3}$
$E[y^2] = \int_0^1 2y^3\,dy = [\frac{1}{2}y^4]_0^1 = \frac{1}{2}$
$Var(y) = E[Y^2] - E[Y]^2 = \frac{1}{2} - \frac{4}{9} = \frac{1}{18}$

(b) $f_{1,2}(x,y) = x e^{-y} I_{(0<x<y<\infty)}$

$f_2(y) = \int_0^y x e^{-y}\,dx = [\frac{1}{2}x^2 e^{-y}]_0^y = \frac{1}{2}y^2 e^{-y} I_{(0,\infty)}$

$f_{1|2}(x|y) = x e^{-y} \div (\frac{1}{2}y^2 e^{-y}) = 2x y^{-2} I_{(0,y)}$

$E[X|Y] = \int_0^y x\, f_{1|2}(x,y)\,dx = \int_0^y 2x^2 y^{-2}\,dx = [\frac{2}{3}x^3 y^{-2}]_0^y = \frac{2}{3}y$

$E[X^2|Y] = \int_0^y 2x^3 y^{-2}\,dx = [\frac{1}{2}x^4 y^{-2}]_0^y = \frac{1}{2}y^2$

$Var(X|Y) = E[X^2|Y] - E[X|Y]^2 = (\frac{1}{2} - \frac{4}{9})y^2 = \frac{1}{18}y^2$

$Var(E[X|Y]) = Var(\frac{2}{3}y) = \frac{4}{9}Var(y)$ ✓✓
$E[Var(X|Y)] = \frac{1}{18}E[y^2]$ ✓✓

$E[y] = \int_0^\infty y\cdot f_2(y)\,dy = \int_0^\infty y\cdot\frac{1}{2}y^2 e^{-y}\,dy = \frac{1}{2}\int_0^\infty y^3 e^{-y}\,dy = \frac{1}{2}\cdot 3! = 3$
$E[y^2] = \int_0^\infty y^2\cdot f_2(y)\,dy = \int_0^\infty \frac{1}{2}y^4 e^{-y}\,dy = \frac{1}{2}\int_0^\infty y^4 e^{-y}\,dy = \frac{1}{2}\cdot 4! = 12$

$\therefore Var(E[X|Y]) = \frac{4}{9}(12-9) = \frac{4}{3}$
$\therefore E[Var(X|Y)] = \frac{1}{18}\cdot 12 = \frac{2}{3}$
$> 4/3 + 2/3 = 2 \cdots *$

$f_X(x) = \int_x^\infty x e^{-y}\,dy = [-x e^{-y}]_x^\infty = x e^{-x} I_{(0,\infty)}$

$E[X] = \int_0^\infty x\cdot f_1(x)\,dx = \int_0^\infty x^2 e^{-x}\,dx = 2! = 2$
$E[X^2] = \int_0^\infty x^2\cdot f_1(x)\,dx = \int_0^\infty x^3 e^{-x}\,dx = 3! = 6$
$\therefore Var(X) = E[X^2] - E[X]^2 = 6 - 2^2 = 2 \cdots = *$

2.11. 확률변수 X와 Y에 대하여 결합확률밀도함수가 다음과 같은 경우에 $Var[Y + X - E[X|Y]]$를 구하여라.

$Cov(X - E[X|Y], v(X)) = 0 \quad \forall v(X)$

$\therefore Var(Y + X - E[X|Y]) = Var(Y) + Var(X - E[X|Y]) \cdots (*)$
$= Var(Y) + E[(X - E[X|Y])^2] - E[X - E[X|Y]]^2$
$= Var(Y) + E[X^2 - 2X\,E[X|Y] + E[X|Y]^2]$
$= Var(Y) + E[X^2] - 2E[X]E[X|Y] + E[X|Y]^2$

* we cannot treat $E[X|Y]$ as a constant. Y is a r.v. here.
$E[2X\,E[X|Y]] = \int_{-\infty}^\infty \int_{-\infty}^\infty 2x\cdot g_1(Y)\,dx\,dy$
function of Y

(*) start here:

recall that $Var(X - E[X|Y]) = E[Var(X|Y)]$

$\therefore Var(Y + X - E[X|Y]) = Var(Y) + E[Var(X|Y)]$

Alternatively:

$Var(Y + X - E[X|Y]) = Var(Y) + Var(X - E[X|Y]) + 2Cov(Y, X - E[X|Y])$
$= Var(Y) + Var(X - E[X|Y])$
$= Var(Y) + Var(X) + Var(E[X|Y]) - 2Cov(X, E[X|Y])$

(a) $f_{1,2}(x,y) = 2 I_{(0<x<y<1)}$

$f_1(x) = \int_x^1 2\,dy = 2(1-x) I_{(0<x<1)}$

$f_2(y) = \int_0^y 2\,dx = 2y I_{(0<y<1)}$

$E[Y] = \int_0^1 2y^2 dy = \frac{2}{3}$, $E[Y^2] = \frac{1}{2}$, $Var(Y) = \frac{1}{2} - (\frac{2}{3})^2 = \frac{1}{18}$

$E[X] = \int_0^1 2x - 2x^2 dx = [x^2 - \frac{2}{3}x^3]_0^1 = 1/3$

$E[X^2] = \int_0^1 2x^2 - 2x^3 dx = [\frac{2}{3}x^3 - \frac{1}{2}x^4]_0^1 = \frac{2}{3} - \frac{1}{2} = 1/6$

$Var(X) = 1/6 - 1/9 = \frac{1}{18}$

$Var(X|Y) = E[X^2|Y] - E[X|Y]^2$

$f_{1|2}(x|y) = f_{1,2}(x,y) / f_2(y) = 2/2y = y^{-1} I_{(0,y)}(x)$

$E[X|Y] = \int_0^y x \cdot y^{-1} dx = [\frac{1}{2}x^2 y^{-1}]_0^y = \frac{1}{2}y$

$E[X^2|Y] = \int_0^y x^2 \cdot y^{-1} dx = [\frac{1}{3}x^3 y^{-1}]_0^y = \frac{1}{3}y^2$

$Var(X|Y) = E[X^2|Y] - E[X|Y]^2 = \frac{1}{3}y^2 - (\frac{1}{2}y)^2 = \frac{1}{12}y^2$

$\therefore E[Var(X|Y)] = E[\frac{1}{12}y^2] = 1/24$

$\therefore Var(Y + X - E[X|Y]) = Var(Y) + E[Var(X|Y)] = \frac{1}{18} + \frac{1}{24} = \frac{4+3}{72} = \frac{7}{72}$

(b) $f_{1,2}(x,y) = 10xy^2 I_{(0<x<y<1)}$

$f_1(x) = \int_x^1 10xy^2 dy = 10x[\frac{1}{3}y^3]_x^1 = \frac{10}{3}x^4 I_{(0,1)}$

$f_2(y) = \int_0^y 10xy^2 dx = [5x^2y^2]_0^y = 5y^4 I_{(0,1)}$

$E[Y] = \int_0^1 5y^5 dy = \frac{5}{6}$ $E[Y^2] = \int_0^1 5y^6 dy = \frac{5}{7}$, $Var(Y) = \frac{5}{7} - (\frac{5}{6})^2 = \frac{5}{7} - \frac{25}{36}$

$f_{1|2}(x,y) = f_{1,2}(x,y)/f_2(y) = 10xy^2 / 5y^4 = 2xy^{-2} I_{(0<x<y<1)}$

$E[X|Y] = \int_0^y 2x^2 y^{-2} dx = \frac{2}{3}y$, $E[X^2|Y] = \int_0^y 2x^3 y^{-2} dx = \frac{1}{2}y^2$

$Var(X|Y) = \frac{1}{2}y^2 - \frac{4}{9}y^2 = \frac{1}{18}y^2$

$E[Var(X|Y)] = \frac{1}{18} E[Y^2] = \frac{1}{18} \cdot \frac{5}{7} = \frac{5}{126}$

$Var(Y + X - E[X|Y]) = Var(Y) + E[Var(X|Y)] = \frac{5}{7} - \frac{25}{36} + \frac{5}{126}$

2.12 1차원 확률변수 Y와 k차원 확률변수 $X = (X_1, \cdots, X_k)^t$에 대하여 $\mu_2 = E[Y]$, $\sigma_{22} = Var(Y)$, $\mu_1 = E[X]$, $\Sigma_{11} = Var(X)$, $\sigma_{21} = Cov(Y, X)$로 나타내고, $\sigma_{22} > 0$이고 Σ_{11}의 역행렬 Σ_{11}^{-1}이 존재할 때 다음에 답하여라.

(a) 임의의 상수 벡터 c와 실수 d에 대하여 다음을 밝혀라.

$$Cov(Y - \mu_2 - \sigma_{21}\Sigma_{11}^{-1}(X - \mu_1),\ c^t X + d) = 0$$

(b) 다차원 확률변수 X의 실수값 1차 함수 $a^t X + b$로서 1차원확률변수 Y를 예측할 때 평균제곱예측오차인

$$E[\{Y - (a^t X + b)\}^2]$$

을 최소로하는 최소제곱선형예측자가 $\mu_2 + \sigma_{21}\Sigma_{11}^{-1}(X - \mu_1)$로 주어지는 것을 밝혀라.

(c) 다음이 성립하는 것을 밝혀라.

$$\max_{a,b} Corr(Y, a^t X + b) = \left(\frac{\sigma_{21}\Sigma_{11}^{-1}\sigma_{12}}{\sigma_{22}}\right)^{1/2} = Corr(Y, \mu_2 + \sigma_{21}\Sigma_{11}^{-1}(X - \mu_1))$$

여기에서 우변의 상관계수를 Y와 $X = (X_1, \cdots, X_k)^t$ 사이의 중상관계수 (multiple correlation coefficient)라고 부르며 $\rho_{Y;X_1,\cdots,X_k}$로 나타낸다.

(d) 다음 등식이 성립하는 것을 밝혀라.

$$Var(Y - \mu_2 - \sigma_{21}\Sigma_{11}^{-1}(X - \mu_1)) = (1 - \rho^2_{Y;X_1,\cdots,X_k})\,\sigma_{22}$$

이러한 등식으로부터 최소제곱선형 예측자 $\mu_2 + \sigma_{21}\Sigma_{11}^{-1}(X - \mu_1)$을 Y분포의 중심 위치로 이용하면 Y의 산포 σ_{22}를 $100\rho^2_{Y;X_1,\cdots,X_k}$(%) 만큼 줄일 수 있다는 뜻이다. 이러한 뜻에서 $\rho^2_{Y;X_1,\cdots,X_k}$를 결정계수 (Coefficient of Determination)라고 부른다.

(Solution)

(a) $Cov(A, B) = E[AB^t] - E[A]E[B]^t$

$\therefore Cov(Y - \mu_2 - \sigma_{21}\Sigma_{11}^{-1}(X - \mu_1),\ c^t X + d)$

let $A = Y - \mu_2 - \sigma_{21}\Sigma_{11}^{-1}(X - \mu_1)$, $B = c^t X + d$

$E[A] = E[Y] - \mu_2 - \sigma_{21}\Sigma_{11}^{-1}(E[X] - \mu_1)$

$\therefore Cov(A, B) = E[AB^t]$

$AB^t = (Y - \mu_2 - \sigma_{21}\Sigma_{11}^{-1}(X - \mu_1))(c^t X + d)^t$

$= YX^t c - \mu_2 X^t c - \sigma_{21}\Sigma_{11}^{-1}(X - \mu_1)X^t c + Ad^t$

$E[AB^t] = E[YX^t]c - \mu_2\mu_1^t c - \sigma_{21}\Sigma_{11}^{-1}(E[XX^t] - \mu_1 E[X]^t)c$

$= (E[YX^t] - E[Y]E[X]^t)c - \sigma_{21}\Sigma_{11}^{-1}Var(X)c$

$= \sigma_{21}c - \sigma_{21}\Sigma_{11}^{-1}\Sigma_{11}c$

$= \sigma_{21}c - \sigma_{21}c = 0$

풀지 못한 문제 해결

(b) 다차원 확률변수 X의 실수값 1차 함수 a^tX+b로서 1차원확률변수 Y를 예측할 때 평균제곱예측오차인

$$E\left[\left\{Y-(a^tX+b)\right\}^2\right]$$

을 최소로하는 최소제곱선형예측자가 $\mu_2+\sigma_{21}\Sigma_{11}^{-1}(X-\mu_1)$로 주어지는 것을 밝혀라.

Let $W=\mu_2+\sigma_{21}\Sigma_{11}^{-1}(X-\mu_1)$ $\quad Cov(Y-W, a^tX+b)=0$, $E[W]=\mu_2=E[Y]$

Let $u(X)=a^tX+b$. $E\left[\left\{Y-(a^tX-b)\right\}^2\right]=E\left[\left\{Y-W+W-u(X)\right\}^2\right]$

$$\begin{aligned}&=E\left[(Y-W)^2+2(Y-W)^t(W-u(X))+(W-u(X))^2\right]\\&=E\left[(Y-W)^2\right]+0+E\left[(W-u(X))^2\right]\\&\geq E\left[(Y-W)^2\right]\end{aligned}$$

$\therefore W=\mu_2+\sigma_{21}\Sigma_{11}^{-1}(X-\mu_1)$ is the linear function of X that minimizes MSE.

$$\begin{aligned}&*\,E[(Y-W)^t(W-u(X))]\\&=E[(Y-W)^t(W-u(X))]-E[(Y-W)]^tE[W-u(X)]\\&=Cov(Y-W, W-u(X))\\&=Cov(Y-W, v(X))=0 \quad \cdots \text{(from 1a)}\end{aligned}$$

(c) 다음이 성립하는 것을 밝혀라.

$$\max_{a,b} Corr(Y, a^tX+b)=\left(\frac{\sigma_{21}\Sigma_{11}^{-1}\sigma_{12}}{\sigma_{22}}\right)^{1/2}=Corr(Y,\mu_2+\sigma_{21}\Sigma_{11}^{-1}(X-\mu_1))$$

여기에서 우변의 상관계수를 Y와 $X=(X_1,\cdots,X_K)^t$ 사이의 중상관계수 (multiple correlation coefficient)라고 부르며 $\rho_{Y;X_1,\cdots,X_K}$로 나타낸다.

$$Corr(Y, a^tX+b)=Cov(Y, a^tX+b)/\left(Var(Y)^tVar(a^tX+b)\right)^{1/2}$$

$Var(Y)=\sigma_{22}$, $Var(a^tX+b)=a^t\Sigma_{11}a$

$\therefore \left(Var(Y)\cdot Var(a^tX+b)\right)^{1/2}=\left(\sigma_{22}\cdot a^t\Sigma_{11}a\right)^{1/2}$

$$\begin{aligned}Cov(Y, a^tX+b)&=Cov(Y, X)a\\&=\sigma_{21}a\end{aligned}$$

$$\begin{aligned}Corr(Y, a^tX+b)&=Cov(Y, X)\cdot a\cdot\left(\sigma_{22}\cdot a^t\Sigma_{11}a\right)^{-1/2}\\&=\sigma_{21}\cdot a\cdot\sigma_{22}^{-1}\left[\left(\Sigma_{11}^{1/2}\cdot a\right)^2\right]^{-1/2}\\&=\frac{\sigma_{21}\cdot a\cdot(\Sigma_{11}^{1/2}\cdot a)^{-1}}{\sigma_{22}}=\frac{\sigma_{21}\cdot a\cdot a^{-1}\cdot\Sigma_{11}^{-1/2}}{\sigma_{22}}=\frac{\sigma_{21}\Sigma_{11}^{-1/2}}{\sigma_{22}}\\&=\frac{\left(\sigma_{21}\Sigma_{11}^{-1}\sigma_{12}\right)^{1/2}}{\sigma_{22}}\end{aligned}$$

(d) 다음 등식이 성립하는 것을 밝혀라.

$$Var(Y-\mu_2-\sigma_{21}\Sigma_{11}^{-1}(X-\mu_1))=(1-\rho^2_{Y;X_1,\cdots,X_K})\,\sigma_{22}$$

이러한 등식으로부터 최소제곱선형예측자 $\mu_2+\sigma_{21}\Sigma_{11}^{-1}(X-\mu_1)$를 Y 분포의 중심 위치로 이용하면 Y의 산포 σ_{22}를 $100\times\rho^2_{Y;X_1,\cdots,X_K}$(%) 만큼 줄일 수 있다는 뜻이다. 이러한 뜻에서 $\rho^2_{Y;X_1,\cdots,X_K}$를 결정계수 (Coefficient of Determination)라고 부른다.

Let $W=\mu_2-\sigma_{21}\Sigma_{11}^{-1}(X-\mu_1)$

$Var(Y-W)=Var(Y)+Var(W)-2Cov(Y,W)$

$$\begin{aligned}Var(W)&=Var(\mu_2-\sigma_{21}\Sigma_{11}^{-1}(X-\mu_1)\\&=Var(-\sigma_{21}\Sigma_{11}^{-1}X)\\&=\sigma_{21}\Sigma_{11}^{-1}Var(X)\Sigma_{11}^{-1}\sigma_{12}\\&=\sigma_{21}\Sigma_{11}^{-1}\sigma_{12}\end{aligned}$$

$$\begin{aligned}Cov(Y,W)&=Cov(Y,X)\Sigma_{11}^{-1}\sigma_{12}\\&=\sigma_{21}\Sigma_{11}^{-1}\sigma_{12}=\rho_{Y;X_1,\cdots,X_K}\end{aligned}$$

$$\begin{aligned}\therefore Var(Y-W)&=\sigma_{22}+\sigma_{21}\Sigma_{11}^{-1}\sigma_{12}-2\sigma_{21}\Sigma_{11}^{-1}\sigma_{12}\\&=\sigma_{22}-\sigma_{21}\Sigma_{11}^{-1}\sigma_{12}\end{aligned}$$

$$\begin{aligned}(1-\rho^2_{Y;X_1,\cdots,X_K})\,\sigma_{22}&=\left(1-\frac{Cov(X,Y)^2}{Var(X)Var(Y)}\right)\sigma_{22}\\&=\sigma_{22}-\frac{\sigma_{12}^2}{\Sigma_{11}}\\&=\sigma_{22}-\sigma_{21}\Sigma_{11}^{-1}\sigma_{12}\end{aligned}$$

$\therefore$ LHS = RHS

2.12 확률변수 X와 Y의 결합 확률밀도함수가 다음과 같은 각 경우에 X와 Y가 서로 독립인가를 판단하고, 독립이 아닌 경우에는 X와 Y의 공분산을 구하여라.

(a) $f_{1,2}(x,y) = 2e^{-x-2y} I_{(x>0, y>0)}$

(b) $f_{1,2}(x,y) = 2e^{-x-y} I_{(0<x<y)}$

(c) $f_{1,2}(x,y) = \frac{25}{104}(1+x^4y^4) I_{(-1\le x\le 1,\ -1\le y\le 1)}$

(d) $f_{1,2}(x,y) = (\frac{1}{6})^x (\frac{1}{3})^y (\frac{1}{2})^{1-x-y}$, $x,y = 0,1$, $x+y = 0,1$

(Solution)

(a) $f_{1,2}(x,y) = e^{-x} I_{(x>0)} \cdot 2e^{-2y} I_{(y>0)} = g_1(x) \cdot g_2(y)$

$\therefore$ X and Y are independent

(b) $f_{1,2}(x,y) = 2e^{-x-y} I_{(0<x<y)}$

$f_1(x) = \int_x^\infty 2e^{-x}e^{-y}dy = 2e^{-x}\left[-e^{-y}\right]_x^\infty = 2e^{-2x} I_{(0\le x)}$

$f_2(y) = \int_0^y 2e^{-x}e^{-y}dx = 2e^{-y}\left[-e^{-x}\right]_0^y = 2e^{-y}(1-e^{-y}) I_{(0<y)}$

$f_{1,2}(x,y) \neq f_1(x) f_2(y)$ $\therefore$ X and Y are not independent.

$M_{X,Y}(t_1,t_2) = E[e^{t_1X+t_2Y}] = \int_0^\infty\int_0^y e^{t_1x+t_2y} \cdot 2e^{-x-y}dx\,dy$

$= 2\int_0^\infty e^{-(1-t_2)y}\int_0^y e^{-(1-t_1)x}dx\,dy$

$= 2\int_0^\infty e^{-(1-t_2)y}\left[-(1-t_1)^{-1}e^{-(1-t_1)y}\right]_0^y dy$

$= 2(1-t_1)^{-1}\int_0^\infty e^{-(1-t_2)y}(1-e^{-(1-t_1)y})dy$

$= 2(1-t_1)^{-1}\int_0^\infty e^{-(1-t_2)y} - e^{-(2-t_1-t_2)y}dy$

$= 2(1-t_1)^{-1}\left[-(1-t_2)^{-1}e^{-(1-t_2)y} + (2-t_1-t_2)^{-1}e^{-(2-t_1-t_2)y}\right]_0^\infty$

$= 2(1-t_1)^{-1}\left[(1-t_2)^{-1} - (2-t_1-t_2)^{-1}\right]$

$= 2\frac{1}{1-t_1}\left\{\frac{2-t_1-t_2-1+t_2}{(1-t_2)(2-t_1-t_2)}\right\}$

$M_{X,Y}(t_1,t_2) = 2(1-t_2)^{-1}(2-t_1-t_2)^{-1}$, $t_2<1$, $t_1+t_2<2$ ✓✓

$M_X(t_1) = M_{X,Y}(t_1, t_2=0) = 2(2-t_1)^{-1}$ ✓✓

$M_Y(t_2) = M_{X,Y}(t_1=0, t_2) = 2(1-t_2)^{-1}(2-t_2)^{-1}$ ✓✓

} verified with ChatGPT 5.0

$M_{X,Y}(t_1,t_2) \neq M_{X_1}(t_1)\cdot M_{X_2}(t_2)$

$\therefore$ X and Y are not independent.

(b) Continued.

$Cov(X,Y) = E[XY] - E[X]E[Y]$

$C_{X,Y}(t_1,t_2) = \log 2(1-t_1)^{-1}(2-t_1-t_2)^{-1}$

$= \log(1-t_1)^{-1}(1-(t_1+t_2)/2)^{-1}$

$= -\log(1-t_1) - \log(1-\frac{t_1+t_2}{2})$

$\frac{\partial}{\partial t_1}\frac{\partial}{\partial t_2}C_{X,Y}(t_1,t_2) = \frac{\partial}{\partial t_1}\frac{1}{2}(1-\frac{t_1}{2}-\frac{t_2}{2})^{-1}$

$= \frac{\partial}{\partial t_1}(2-t_1-t_2)^{-1}$

$= (2-t_1-t_2)^{-2}$

$C_{1,1} = \frac{\partial}{\partial t_1}\frac{\partial}{\partial t_2}C_{X,Y}(t_1=0, t_2=0)$

$= 2^{-2} = 1/4$

$Cov(X,Y) = 1/4$ ✓✓ * try using pdf and mgf for exercise!!

(c) $f_{1,2}(x,y) = \frac{25}{104}(1+x^4y^4) I_{(-1\le x\le 1,\ -1\le y\le 1)}$

$f_1(x) = \frac{25}{104}\int_{-1}^1 1+x^4y^4dy = \frac{25}{104}\left[y+\frac{1}{5}x^4y^5\right]_{-1}^1 = \frac{25}{104}(2+\frac{2}{5}x^4)$

$f_2(y) = \frac{25}{104}\int_{-1}^1 1+x^4y^4dx = (\cdots) = \frac{25}{104}(2+\frac{2}{5}y^4)$

X and Y are not independent.

$M_{X,Y}(s,t) = E[e^{sX+tY}] = \int_{-1}^1\int_{-1}^1 e^{sx+ty}\frac{25}{104}(1+x^4y^4)dx\,dy.$

(calculation is too lengthy.)

$Cov(X,Y) = E[XY] - E[X]E[Y]$

$E[XY] = \frac{25}{104}\int_{-1}^1\int_{-1}^1 xy + x^5y^5\,dx\,dy = \frac{25}{104}\int_{-1}^1\left[\frac{1}{2}x^2y + \frac{1}{6}x^6y^5\right]_{-1}^1 dy$

$= \frac{25}{104}\int_{-1}^1 0\,dy = 0$

$E[X] = \frac{25}{104}\int_{-1}^1 2x + \frac{2}{5}x^5dx = \frac{25}{104}\left[x^2 + \frac{2}{30}x^6\right]_{-1}^1 = 0$

$E[Y] = (\cdots) = 0$

$\therefore Cov(X,Y) = E[XY] - E[X]E[Y] = 0 - 0\cdot 0 = 0$

(d) $f_{1,2}(x,y) = (1/6)^x(1/3)^y(1/2)^{1-x-y}$, $x,y = 0,1$, $x+y = 0,1$

$A := \{(x,y) : x,y = 0 \text{ or } 1,\ x+y = 0,1\}$

$= \{(0,0), (0,1), (1,0)\}$

$f_1(x) = \begin{cases} x=0 \text{ 제외} \\ x=1 \text{ 제외} \end{cases} = \begin{cases} \text{if } x=0,\ f_1(x) = f_{1,2}(0,0) + f_{1,2}(0,1) \\ \text{if } x=1,\ f_1(x) = f_{1,2}(1,0) \end{cases}$

$= \begin{cases} (1/6)^0(1/3)^0(1/2)^1 + (1/6)^0(1/3)^1(1/2)^0 = \frac{5}{6}, & x=0 \\ (1/6)^1(1/3)^0(1/2)^0 = 1/6, & x=1 \end{cases}$

$$f_2(y) = \begin{cases} y=0 \text{ 경우} \\ y=1 \text{ 경우} \end{cases} = \begin{cases} (1/6)^0(1/3)^0(1/2)^1 + (1/6)^1(1/3)^0(1/2)^0 = 2/3 \\ (1/6)^0(1/3)^1(1/2)^0 = 1/3 \end{cases}$$

$$f_1(x) = \frac{5}{6} I_{(x=0)} + \frac{1}{6} I_{(x=1)}$$

$$f_2(y) = \frac{2}{3} I_{(y=0)} + \frac{1}{3} I_{(y=1)}$$

$$f_1(x) f_2(y) = \frac{10}{18} I_{(x=0,y=0)} + \frac{1}{9} I_{(x=1,y=0)} + \frac{5}{18} I_{(x=0,y=1)} + \frac{1}{18} I_{(x=1,y=1)}$$

$\therefore f_{1,2}(x,y) \neq f_1(x) f_2(y)$ $\therefore$ X and Y are not independent.

$E[XY] = P(X=1, Y=1) = 0$

$E[X] = P(X=1) = 1/6$ $E[Y] = P(Y=1) = 1/3$

$\therefore \text{Cov}(X,Y) = 0 - 1/6 \cdot 1/3 = -1/18$ ✓✓

2.14. 확률변수 X_1, X_2, X_3 의 결합확률밀도함수가

$$f_{1,2}(x,y) = 6e^{-x-2y} I_{(0<x<y<\infty)}$$

일 때 다음에 답하여라.

(a) X와 Y의 결합적률생성함수 $mgf_{1,2}(t_1, t_2)$를 구하여라.

(b) X+Y 의 분산을 구하여라

(c) X와 Z=Y-X 의 결합적률생성함수 $M_{X,Z}(s,t)$를 구하고, X와 Z가 서로 독립인가를 판단하여라.

(a) $M_{1,2}(t_1, t_2) = E[e^{t_1X + t_2Y}] = \int_0^\infty \int_0^y e^{t_1x + t_2y} \cdot 6e^{-x-2y} \, dx\,dy$

$= 6\int_0^\infty e^{-(2-t_2)y} \int_0^y e^{-(1-t_1)x} \, dx\,dy$

$= 6\int_0^\infty e^{-(2-t_2)y} \left[-(1-t_1)^{-1} e^{-(1-t_1)x} \right]_0^y dy$

$= 6(1-t_1)^{-1} \int_0^\infty e^{-(2-t_2)y} \left(1 - e^{-(1-t_1)y}\right) dy$

$= 6(1-t_1)^{-1} \int_0^\infty e^{-(2-t_2)y} - e^{-(3-t_1-t_2)y} \, dy$

$= 6(1-t_1)^{-1} \left[-(2-t_2)^{-1} e^{-(2-t_2)y} + (3-t_1-t_2)^{-1} e^{-(3-t_1-t_2)y} \right]_0^\infty$

$= 6(1-t_1)^{-1} \left\{ (2-t_2)^{-1} - (3-t_1-t_2)^{-1} \right\}$

$= 6 \frac{1}{1-t_1} \left[\frac{1}{2-t_2} - \frac{1}{3-t_1-t_2} \right]$

$= 6 \frac{1}{1-t_1} \frac{1-t_1}{(2-t_2)(3-t_1-t_2)} = \left(1 - \frac{t_2}{2}\right)^{-1} \left(1 - \frac{t_1+t_2}{3}\right)^{-1}$

$t_2 < 2,\ t_1 + t_2 < 3$

(b) $\text{Var}(Y+X) = \text{Var}(Y) + \text{Var}(X) + 2\,\text{Cov}(X,Y)$

$C_{1,2}(t_1, t_2) = \log M_{1,2}(t_1, t_2) = -\log\left(1 - \frac{t_2}{2}\right) - \log\left(1 - \frac{t_1+t_2}{3}\right)$, $|t_1|<2$, $|t_1+t_2|<3$

$\frac{\partial}{\partial t_1} C_{1,2}(t_1, t_2) = \frac{1}{3} \frac{1}{1-(t_1+t_2)/3} = \frac{1}{3-t_1-t_2}$

$\frac{\partial}{\partial t_2} \frac{\partial}{\partial t_1} C_{1,2}(t_1, t_2) = (3-t_1-t_2)^{-2}$ $\therefore \text{Cov}(X,Y) = 1/9$

$\frac{\partial^2}{\partial t_1^2} C_{1,2}(t_1, t_2) = (3-t_1-t_2)^{-2}$ $\text{Var}(X) = \frac{\partial^2}{\partial t_1^2} C_{1,2}(t_1=0, t_2=0) = \frac{1}{18}$ $\frac{1}{9}$

$\frac{\partial^2}{\partial t_2^2} C_{1,2}(t_1, t_2) = (2-t_2)^{-2} + (3-t_1-t_2)^{-2}$; $\text{Var}(Y) = \frac{\partial^2}{\partial t_2^2} C_{1,2}(t_1=0, t_2=0) = \frac{1}{2}\left(\frac{1}{4} + \frac{1}{9}\right)$

$= \frac{1}{2} \frac{4+9}{36} = 13/72$ $\frac{13}{36}$

$\therefore \text{Var}(X+Y) = \text{Var}(X) + \text{Var}(Y) + 2\,\text{Cov}(X,Y) = \frac{1}{9} + \frac{2}{9} + \frac{13}{36} = \frac{4+8+13}{36} = \frac{25}{36}$ ✓✓

(c) $M_{X,Z}(s,t) = E[e^{sX+tZ}] = E[e^{sX+t(Y-X)}]$

$= E[e^{(s-t)X + tY}]$

$t_1 = s-t$, $t_2 = t$

$\therefore M_{X,Z}(s,t) = \left(1 - \frac{t}{2}\right)^{-1} \left(1 - \frac{s}{3}\right)^{-1}$ $t<2, s<3$

$M_X(s) = M_{X,Z}(s, t=0) = \left(1 - \frac{s}{3}\right)^{-1}$, $M_Z(t) = M_{X,Z}(s=0, t) = \left(1 - \frac{t}{2}\right)^{-1}$

$\therefore M_{X,Z}(s,t) = M_X(s) \cdot M_Z(t)$ $\therefore$ Z and X is independent

① cgf 에서 t 의 구간을 적을 때 절대값을 써라.

이유: cgf 는 $-\log(1-A)$ 의 멱급수 전개해 같다. $-\log(1-A)$ 의 멱급수 전개에서 Taylor series 의 조건으로 0 을 포함한 구간 안에 A가 있어야 함으로 $-1<A<1 \Rightarrow |A|<1$ 열린

② cgf 를 미분하여 $t_1=0, t_2=0$ 을 대입해 분산, 공분산을 얻을 때 $\frac{1}{2}$ 을 곱하지 않아도 된다.

이유: cgf 를 멱급수 전개하여 미분했을 때 분모항의 $0!$ 이 Cancel out 되는 것을 볼 수 있다.

하지만, 멱급수 전개식에서 t_1t_2, t_1^2 또는 t_2^2 의 계수 (coefficient) 를 활용해 분산, 공분산을 계산할 경우 분모항인 $0!$ 에 유의하여야 한다.

2.15. 확률변수 X_1, X_2, X_3 의 결합확률밀도함수가

$$f_{1,2,3}(x_1, x_2, x_3) = 48 x_1 x_2 x_3 I_{(0<x_1<x_2<x_3<1)}$$

일 때 다음에 답하여라.

(a) $X_3 = z$ 인 조건에서, X_1과 X_2의 조건부확률밀도함수 $f_{1,2|3}(x_1, x_2|z)$를 구하여라.

(b) 세 확률변수 X_1, X_2, X_3는 서로 독립인가를 판단하여라.

[Solution]

(a) $f_3(x_3) = \int_0^{x_3}\int_0^{x_2} 48 x_1 x_2 x_3 \, dx_1 dx_2 = \int_0^{x_3} \left[24 x_1^2 x_2 x_3\right]_0^{x_2} dx_2$

$= \int_0^{x_3} 24 x_2^3 x_3 \, dx_2 = \left[6 x_2^4 x_3\right]_0^{x_3} = 6 x_3^5 I_{(0<x_3<1)}$

$f_{1,2|3}(x_1, x_2|x_3) = f_{1,2,3}(x_1, x_2, x_3) / f_3(x_3)$

$= 8 x_1 x_2 x_3^{-4} I_{(0<x_1<x_2<x_3<1)}$ ✓✓

$f_{1,2|3}(x_1, x_2|z) = 8 x_1 x_2 z^{-4} I_{(0<x_1<x_2<z)}$

(b) $M_{1,2,3}(s,t,u) = E\left[\exp\{s x_1 + t x_2 + u x_3\}\right]$

$= \int_0^1\int_0^{x_3}\int_0^{x_2} e^{s x_1 + t x_2 + u x_3} \cdot 48 x_1 x_2 x_3 \, dx_1 dx_2 dx_3$

[Calculation is lengthy]

$f_1(x_1) = \int_{x_1}^1\int_{x_2}^1 48 x_1 x_2 x_3 \, dx_3 dx_2 = \int_{x_1}^1 \left[24 x_1 x_2 x_3^2\right]_{x_2}^1 dx_2$

$= \int_{x_1}^1 24 x_1 x_2 - 24 x_1 x_2^3 \, dx_2$

$= \left[12 x_1 x_2^2 - 6 x_1 x_2^4\right]_{x_1}^1$

$= (12 x_1 - 6 x_1) - (12 x_1^3 - 6 x_1^5)$

$= 6 x_1 - 12 x_1^3 + 6 x_1^5 I_{(0<x_1<1)}$

$= 6(x_1 - 2x_1^3 + x_1^5) I_{(0<x_1<1)}$

$f_2(x_2) = \int_{x_2}^1\int_0^{x_2} 48 x_1 x_2 x_3 \, dx_1 dx_3 = \int_{x_2}^1 \left[24 x_1^2 x_2 x_3\right]_0^{x_2} dx_3$

$= \int_{x_2}^1 24 x_2^3 x_3 \, dx_3 = \left[12 x_2^3 x_3^2\right]_{x_2}^1$

$= 12 x_2^3 - 12 x_2^5$

$= 12 x_2^3 (1 - x_2^2) I_{(0<x_2<1)}$

$f_{1,2,3}(x_1, x_2, x_3) \neq f_1(x_1) f_2(x_2) f_3(x_3)$ $\therefore$ X_1, X_2 and X_3 are not independent. ✓✓

2.16 확률변수 X_1, X_2, X_3 의 결합확률밀도함수가

$$f_{1,2,3}(x,y,z) = 90 e^{-(x+2y+3z)} I_{(0<x<y<z<\infty)}$$

일 때 다음에 답하여라.

(a) 세 확률변수 X_1, X_2, X_3의 결합적률생성함수 $mgf_{1,2,3}(t_1, t_2, t_3)$를 구하여라.

(b) $X = (X_1, X_2, X_3)'$의 분산행렬 $Var(X)$를 구하여라.

(c) $X_1 + X_2 + X_3$ 의 분산을 구하여라.

(d) $X_1, X_2 - X_1, X_3 - X_2$ 가 서로 독립인가를 판단하여라.

(a) $E[e^{t'X}] = E[e^{t_1 x_1 + t_2 x_2 + t_3 x_3}]$

$Y_1 = X_1$ $\quad X_1 = Y_1$

$Y_2 = X_2 - X_1$ $\quad X_2 = Y_2 + Y_1$ $\quad |J_{x'}| = \begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{vmatrix} = 1$

$Y_3 = X_3 - X_2$ $\quad X_3 = Y_3 + Y_2 + Y_1$

$f_{1,2,3}(x_1, x_2, x_3) = 90 \exp\{-(y_1 + 2(y_1+y_2) + 3(y_1+y_2+y_3))\} I_{(0<y_1<y_2<y_3<\infty)}$

$= 90 \exp\{-(6y_1 + 5y_2 + 3y_3)\} I_{(0<y_1<y_2<y_3<\infty)}$

$E[e^{tX}] = \int_0^\infty\int_0^z\int_0^y e^{t_1 x + t_2 Y + t_3 Z} \, 90 \exp\{-x - 2Y - 3Z\} \, dx \, dY \, dz$

$= \int_0^\infty\int_x^\infty\int_y^\infty 90 \exp\{-(1-t_1)x - (2-t_2)Y - (3-t_3)z\} dz dy dx$

$= 90 \int_0^\infty \exp\{-(1-t_1)x\} \int_x^\infty \exp\{-(2-t_2)Y\} \int_y^\infty \exp\{-(3-t_3)z\} dz dy dx$

$= 90 \int_0^\infty \exp\{-(1-t_1)x\} \int_x^\infty \exp\{-(2-t_2)Y\} \left[-\frac{1}{3-t_3} \exp\{-(3-t_3)z\}\right]_y^\infty dy dx$

$= 90 \cdot (3-t_3)^{-1} \int_0^\infty \exp\{-(1-t_1)x\} \int_x^\infty \exp\{-(5-t_2-t_3)y\} \, dy \, dx$

$= 90 \cdot (3-t_3)^{-1} \int_0^\infty \exp\{-(1-t_1)x\} \left[-\frac{1}{5-t_2-t_3} \exp\{-(5-t_2-t_3)y\}\right]_x^\infty$

$= 90 (3-t_3)^{-1} (5-t_2-t_3)^{-1} \int_0^\infty \exp\{-(6-t_1-t_2-t_3)x\} dx$

$= 90 (3-t_3)^{-1} (5-t_2-t_3)^{-1} (6-t_1-t_2-t_3)^{-1}$ $\quad t_3 < 3,\ t_2+t_3 < 5,\ t_1+t_2+t_3 < 6$

$= (1 - t_3/3)^{-1} (1 - \frac{t_2+t_3}{5})^{-1} (1 - \frac{t_1+t_2+t_3}{6})^{-1}$

(b) $C_X(t) = \log(M_X(t)) = -\log(1 - \frac{t_3}{3}) - \log(1 - \frac{t_2+t_3}{5}) - \log(1 - \frac{t_1+t_2+t_3}{6})$

$= (\frac{t_3}{3}) + \frac{1}{2}(\frac{t_3}{3})^2 + \cdots + (\frac{t_2+t_3}{5}) + \frac{1}{2}(\frac{t_2+t_3}{5})^2 + \cdots + (\frac{t_1+t_2+t_3}{6}) + \frac{1}{2}(\frac{t_1+t_2+t_3}{6})^2 +$

$\frac{\partial}{\partial t_1} C_X(t=0) = \frac{1}{6}$

$\frac{\partial}{\partial t_2} C_X(t) = \frac{1}{5}$

$\frac{\partial}{\partial t_3} C_X(t=0) = \frac{1}{3} + \frac{1}{5} + \frac{1}{6}$

$\frac{\partial^2}{\partial t_1 \partial t_2} C_X(t)\Big|_{t=0} = \frac{1}{36}$ $\quad \frac{\partial^2}{\partial t_1^2} C_X(t)\Big|_{t=0} = \frac{1}{36}$

$\frac{\partial^2}{\partial t_1 \partial t_3} C_X(t)\Big|_{t=0} = \frac{1}{36}$ $\quad \frac{\partial^2}{\partial t_2^2} C_X(t)\Big|_{t=0} = \frac{1}{25} + \frac{1}{36}$

$\frac{\partial^2}{\partial t_2 \partial t_3} C_X(t)\Big|_{t=0} = \frac{1}{25} + \frac{1}{36}$ $\quad \frac{\partial^2}{\partial t_3^2} C_X(t)\Big|_{t=0} = \frac{1}{9} + \frac{1}{25} + \frac{1}{36}$

$$\therefore \mathrm{Var}(X) = \begin{bmatrix} 1/36 & 1/36 & 1/36 \\ 1/36 & 1/25+1/36 & 1/25+1/36 \\ 1/36 & 1/25+1/36 & 1/4+1/25+1/36 \end{bmatrix}$$ ✓✓

(c) $\mathrm{Var}(X_1+X_2+X_2) = \mathrm{Var}(\mathbb{1}^t X) = \mathbb{1}^t \mathrm{Var}(X)\mathbb{1}$

$$= \begin{bmatrix} 3/36 & 2/25+3/36 & 1/4+2/25+3/36 \end{bmatrix}\mathbb{1}$$

$$= 1/4+4/25+9/36$$ ✓✓

(Approach #1)

(d) $f_{1,2,3}(x_1,x_2,x_3) = 90\exp\{-(y_1+2(y_1+y_2)+3(y_1+y_2+y_3)\}\, I_{(0,+\infty)}^{(y_1,y_2,y_3)}$

$= 90\exp\{-(6y_1+5y_2+3y_3)\}\, I_{(0,+\infty)}^{(y_1,y_2,y_3)}$

$= 6\exp\{-6y_1\} I_{(0,+\infty)}^{(y_1)} \cdot 5\exp\{-5y_2\} I_{(0,\infty)}^{(y_2)} \cdot 3\exp\{-3y_3\} I_{(0,\infty)}^{(y_3)}$

$\therefore X_1$, X_2-X_1 and X_3-X_2 are independent.

(Approach #2)

$E[\exp\{t^tY\}] = E[e^{t_1X_1+t_2(X_2-X_1)+t_3(X_3-X_2)}]$

$= E[e^{(t_1-t_2)X_1+(t_2-t_3)X_2+t_3X_3}]$

$= E[e^{sX_1+uX_2+t_3V_3}]$, $t_1-t_2=s$ and $t_2-t_3=u$

$=$

$= (1-t_3/3)^{-1}(1-\frac{u+t_3}{5})^{-1}(1-\frac{s+u+t_3}{6})^{-1}$

$= (1-t_3/3)^{-1}(1-\frac{t_2}{5})^{-1}(1-\frac{t_1}{6})^{-1}$

$= g(t_3)\cdot g(t_2)\cdot g(t_1)$

$\therefore X_1$, X_2-X_1 and X_3-X_2 are independent. ✓✓

Q.17. 확률변수 X_1, X_2, X_3의 결합확률밀도함수가

$$f_{1,2,3}(x,y,z) = 90e^{-(x+2y+3z)} I_{(0<x<y<z<+\infty)}$$

일 때 다음에 답하여라.

(a) $X_1 = x$인 조건에서, X_2와 X_3의 조건부확률밀도함수 $f_{2,3|1}(x_2,x_3|x)$를 구하여라.

(b) $Y=(X_2,X_3)^t$ 라고 할 때 $E[Y|X_1]$, $\mathrm{Var}(Y|X_1)$를 구하여라.

(c) $Y=(X_2,X_3)^t$ 라고 할 때 $\mathrm{Var}[E[Y|X]]$, $E[\mathrm{Var}[Y|X]]$를 구하여라.

$$f_1(x) = \int_x^\infty \int_y^\infty 90e^{-(x+2y+3z)}\,dz\,dy$$

$$= \int_x^\infty \left[-30e^{-(x+2y+3z)}\right]_y^\infty dy$$

$$= \int_x^\infty 30e^{-(x+5y)}dy = \left[-6e^{-(x+5y)}\right]_x^\infty = 6e^{-6x} I_{(0<x<+\infty)}^{(x)}$$ ✓✓

$$f_{2,3|1}(y,z|x) = 15e^{-(2y+3z-5x)} I_{(0<x<y<z<+\infty)}$$ ✓✓

$$E[Y|X_1] = \begin{pmatrix} E[X_2|X_1] \\ E[X_3|X_1] \end{pmatrix}$$

$$E[X_2|X_1] = \int_x^{+\infty}\int_y^{+\infty} y\cdot 15e^{-(2y+3z-5x)}\,dz\,dy$$

$$= \int_x^{+\infty}\left[-5ye^{-(2y+3z-5x)}\right]_y^{+\infty} dy$$

$$= \int_x^{+\infty} 5ye^{-(5y-5x)}dy = \left[-ye^{-(5y-5x)}\right]_x^{+\infty} - \int_x^{+\infty} -e^{-(5y-5x)}dy$$

$$= \left[0+x\right] - \left[\tfrac{1}{5}e^{-(5y-5x)}\right]_x^{+\infty}$$

$$= x+\tfrac{1}{5}$$ ✓✓

$$E[X_3|X_1] = \int_x^\infty \int_x^z z\cdot 15e^{-(2y+3z-5x)}\,dy\,dz$$

$$= \int_x^{+\infty}\left[z\cdot\tfrac{15}{2}e^{-(2y+3z-5x)}\right]_x^z dz$$

$$= \int_x^{+\infty}\left[z\cdot\tfrac{15}{2}e^{-(5z-5x)} - z\cdot\tfrac{15}{2}e^{-(3z-3x)}\right]dz$$

$$= \cdots$$

[Textbook Approach]

(2.7)과 같은 방법으로 $t_3/3<1$, $(t_2+t_3)/5<1$ 일 때

$$cgf_{2,3|1}(t_2,t_3|x) = -\log(1-t_3/3) - \log(1-(t_2+t_3)/5) + (t_2+t_3)x$$

이로부터 $-\log(1-A) = A+\frac{1}{2}A^2+\frac{1}{3}A^3+\cdots$, $-1<A<1$ 을 이용하여 두물생성함수를 전개하고 일차항과 이차항의 계수를 찾으면 평균벡터와 분산행렬을 구할 수 있다.

$$E[Y|X_1] = \begin{pmatrix} X_1+1/5 \\ X_1+8/15 \end{pmatrix},\quad \mathrm{Var}(Y|X_1) = \begin{pmatrix} 1/25 & 1/25 \\ 1/25 & 1/25+1/9 \end{pmatrix}$$: 대칭행렬

(c) $\mathrm{Var}[E[Y|X_1]] = \mathrm{Var}\begin{bmatrix} X_1+1/5 \\ X_1+8/15 \end{bmatrix} = \begin{pmatrix} 1/36 & 1/36 \\ 1/36 & 1/36 \end{pmatrix}$: 대칭행렬

$$E[\mathrm{Var}(Y|X_1)] = \begin{bmatrix} 1/25 & 1/25 \\ 1/25 & 1/25+1/9 \end{bmatrix}$$: 대칭행렬