4.1. 균등분포 $U(-\pi/2, +\pi/2)$를 따르는 확률변수 X에 대하여 $Y=\tan(X)$의 확률밀도함수를 구하여라.

$\mathcal{X} = \{x : -\frac{\pi}{2} < x < \frac{\pi}{2}\}$, $\mathcal{Y} := \{y : -\infty < y < +\infty\}$

$u : \mathcal{X} \mapsto \mathcal{Y}$, $u(x) = \tan(x)$
$u^{-1} : \mathcal{Y} \mapsto \mathcal{X}$, $u^{-1}(y) = \tan^{-1}(x)$

$\frac{\partial \tan(x)}{\partial x} = \frac{\cos(x)}{\cos(x)} + \frac{\sin(x)^2}{\cos(x)^2} = 1 + \tan^2(x)$

1. domain and range of transformation is an open interval in $\mathbb{R}$
2. transformation is 1-to-1
3. $|J_{u^{-1}}| = |\partial x/\partial y^{-1}| = |(1+\tan^2(y)|^{-1}| = |(1+y^2)^{-1}| = \frac{1}{1+y^2}$

$$f_X(x) = f_Y(u^{-1}(y)) \cdot |J_{u^{-1}}| = \frac{1}{\pi/2 - (-\pi/2)} \cdot I_{(-\infty,+\infty)}(y) \frac{1}{1+y^2}$$
$$= \{\pi(1+y^2)\}^{-1} I_{(-\infty,+\infty)}(y)$$

4.2. 확률변수 X_1, X_2의 결합확률밀도함수가

$$f(x_1, x_2) = 2e^{-x_1 - x_2} I_{(0<x_1<x_2)}$$

일 때 $Y_1 = X_1$, $Y_2 = X_2 - X_1$의 결합확률밀도함수를 구하고, Y_1과 Y_2가 서로 독립인가를 판단하여라.

$\mathcal{X} = \{(x_1, x_2) : 0 < x_1 < x_2\}$, $\mathcal{Y} := \{(y_1, y_2) : 0 < y_1 < +\infty, 0 < y_2 < +\infty\}$

$$u(x) \begin{cases} Y_1 = X_1 \\ Y_2 = X_2 - X_1 \end{cases}, \quad u^{-1}(Y) \begin{cases} X_1 = Y_1 \\ X_2 = Y_1 + Y_2 \end{cases}$$

∴ transformation u is one-to-one and the range and domain are open ~~spaces~~ set defined in $\mathbb{R}^2$

$$|J_{u^{-1}}| = \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1$$

$$f_X(x) = f_Y(u^{-1}(x)) |J_{u^{-1}}| = 2e^{-2y_1 - y_2} I_{(0<y_1<+\infty)} I_{(0<y_2<+\infty)}$$
$$= \left[2e^{-2y_1} I_{(0,+\infty)}(y_1)\right]\left[e^{-y_2} I_{(0,+\infty)}(y_2)\right] = f_Y(y_1, y_2), \quad Y = (Y_1, Y_2)^t$$

∴ Y_1 and Y_2 are independent. ✓✓

4.3. 확률변수 X_1, X_2의 결합확률밀도함수가

$$f(x_1, x_2) = 8 x_1 x_2 I_{(0<x_1<x_2<1)}$$

일 때 $Y_1 = X_1/X_2$, $Y_2 = X_2$의 결합확률밀도함수를 구하고, Y_1과 Y_2가 서로 독립인가를 판단하여라.

$\mathcal{X} := \{(x_1, x_2) : 0 < x_1 < x_2 < 1, x_1, x_2 \in \mathbb{R}\}$
$\mathcal{Y} := \{(y_1, y_2) : 0 < y_1 < 1, 0 < y_2 < 1\}$

$$u(x) \begin{cases} Y_1 = X_1/X_2 \\ Y_2 = X_2 \end{cases} \quad u^{-1}(Y) \begin{cases} X_1 = Y_1 Y_2 \\ X_2 = Y_2 \end{cases}$$

$$|J_{u^{-1}}| = |\det(\frac{\partial x}{\partial y})| = \begin{vmatrix} y_2 & y_1 \\ 0 & 1 \end{vmatrix} = y_2$$

(1) Transformation u is 1-1
(a) domain & range of u is an open set defined in $\mathbb{R}^2$.
(b) $|J_{u^{-1}}| \neq 0$

$$\therefore f_X(x) = f_Y(u^{-1}(x_1, x_2)) |J_{u^{-1}}| = 8 y_1 y_2 \cdot y_2 \cdot I_{(0,1)}(y_1) \cdot I_{(0,1)}(y_2)\, y_2$$
$$= \left[2y_1 I_{(0,1)}(y_1)\right]\left[4y_2^3 I_{(0,1)}(y_2)\right]$$

∴ Y_1 and Y_2 are independent. ✓✓

4.4. 확률변수 X_1, X_2, X_3의 결합확률밀도함수가

$$f(x_1, x_2, x_3) = 6e^{-x_1 - x_2 - x_3} I_{(0 \le x_1 \le x_2 \le x_3)}$$

일 때 $Y_1 = X_1$, $Y_2 = X_2 - X_1$, $Y_3 = X_3 - X_2$의 결합확률밀도함수를 구하고, Y_1, Y_2, Y_3가 서로 독립인가를 판단하여라.

$\mathcal{X} := \{(x_1, x_2, x_3) : 0 \le x_1 \le x_2 \le x_3\}$
$\mathcal{Y} := \{(y_1, y_2, y_3) : 0 \le y_1 < +\infty, 0 \le y_2 < +\infty, 0 \le y_3 < +\infty\}$

$$u(x) : \begin{cases} Y_1 = X_1 \\ Y_2 = X_2 - X_1 \\ Y_3 = X_3 - X_2 \end{cases} \quad u^{-1}(Y) : \begin{cases} X_1 = Y_1 \\ X_2 = Y_2 + Y_1 \\ X_3 = Y_3 + Y_2 + Y_1 \end{cases}$$ Also domain is an open set

$$|J_{u^{-1}}| = |\det(\frac{\partial x}{\partial y})| = \begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{vmatrix} = 1 \neq 0$$

$$f_X(x) = f_Y(u^{-1}(x)) |J_{u^{-1}}| = 6e^{-(3y_1 + 2y_2 + y_3)} \cdot 1 \cdot I_{[0,1]}(y_1) \cdot I_{[0,1]}(y_2) \cdot I_{[0,1]}(y_3)$$
$$= \left[3e^{-3y_1} I_{[0,+\infty)}(y_1)\right]\left[2e^{-2y_2} I_{[0,+\infty)}(y_2)\right]\left[e^{-y_3} I_{[0,+\infty)}(y_1)\right]$$

∴ Y_1, Y_2 and Y_3 are independent. ✓✓

* The range of the function u need not be open

4.5. 확률변수 X_1, X_2, X_3가 서로 독립이고 각각의 주변확률밀도함수가

$$f(x) = e^{-x} I_{(0,\infty)}(x)$$

일 때 다음에 답하여라.

(a) $Y_1 = X_1$, $Y_2 = X_1 + X_2$, $Y_3 = X_1 + X_2 + X_3$의 결합확률밀도함수를 구하여라.

$$\mathcal{X} := \{ (x_1, x_2, x_3) : 0 < x_1 < +\infty, 0 < x_2 < +\infty, 0 < x_3 < +\infty \}$$
$$\mathcal{Y} := \{ (y_1, y_2, y_3) : 0 < y_1 < y_2 < y_3 \}$$

$$u(x) \begin{cases} Y_1 = X_1 \\ Y_2 = X_1 + X_2 \\ Y_3 = X_1 + X_2 + X_3 \end{cases}, \quad u^{-1}(Y) \begin{cases} X_1 = Y_1 \\ X_2 = Y_2 - Y_1 \\ X_3 = Y_3 - Y_2 \end{cases}$$

$$|J_{u^{-1}}| = \left| \det\left(\frac{\partial x}{\partial y}\right) \right| = \begin{vmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{vmatrix} = 1$$

(1) $u(x)$ is one-to-one
(2) domain $\mathcal{X}$ is an open set in $\mathbb{R}^3$ + $u(x)$ is continously differentiable
(3) $|J_{u^{-1}}| \neq 0$

$$f_{X_1,X_2,X_3}(x_1,x_2,x_3) = e^{-x_1-x_2-x_3} I_{(0,+\infty)}(x_1) I_{(0,+\infty)}(x_2) I_{(0,+\infty)}(x_3)$$

$$f_{Y_1,Y_2,Y_3}(y_1,y_2,y_3) = f_{X_1,X_2,X_3}(u^{-1}(x)) = e^{-y_3} I_{(0<y_1<y_2<y_3)}$$

(b)
$$f_{Y_1,Y_3}(y_1,y_3) = \int_{y_1}^{y_3} e^{-y_3} I_{(0,+\infty)}(y_3) I_{(0,+\infty)}(y_1) dy_2 = (y_3 - y_1) e^{-y_3} I_{(0<y_1<y_3)}$$ ✓✓

4.6. 확률변수 X_1, X_2의 결합확률밀도함수가

$$f_{1,2}(x_1,x_2) = 2e^{-x_1-x_2} I_{(0<x_1<x_2<\infty)}$$

이고, $Y_1 = X_1/X_2$, $Y_2 = X_2$라고 할 때 다음에 답하여라.

(a) Y_1의 확률밀도함수를 구하여라.
(b) $E[Y_2|Y_1]$과 $Var(Y_2|Y_1)$을 구하여라.

(a)

$$\mathcal{X} = \{ (x_1,x_2) : 0 < x_1 < x_2 < +\infty \}$$
$$\mathcal{Y} = \{ (y_1,y_2) : 0 < y_1 < 1, 0 < y_2 < +\infty \}$$

$$u(x) \begin{cases} Y_1 = X_1/X_2 \\ Y_2 = X_2 \end{cases} \quad u^{-1}(Y) \begin{cases} X_1 = Y_1 Y_2 \\ X_2 = Y_2 \end{cases}$$

$$|J_{u^{-1}}| = \left| \det\left(\frac{\partial x}{\partial y}\right) \right| = \begin{vmatrix} y_2 & y_1 \\ 0 & 1 \end{vmatrix} = y_2$$

(1) $\mathcal{X}$ is an open set defined in $\mathbb{R}^2$
(2) $u : \mathcal{X} \mapsto \mathcal{Y}$ is 1-1
(3) $|J| \neq 0$

$$\therefore f_X(x) = f_Y(u^{-1}(x)) |J_{u^{-1}}|$$
$$= 2e^{-y_1 y_2} e^{-y_2} y_2 I_{(0<y_1<1)} I_{(0<y_2<+\infty)}$$

$$f_{Y_1}(y_1) = \int_0^\infty 2y_2 e^{-y_2(1+y_1)} dy_2 =$$
$$= 2\left\{ \left[-(1+y_1)^{-1} y_2 e^{-y_2(1+y_1)} \right]_0^\infty - \int_0^\infty -(1+y_1)^{-1} e^{-1} e^{-y_2(1+y_1)} dy_2 \right\}$$
$$= 2\int_0^\infty (1+y_1)^{-1} e^{-(1+y_1)y_2} dy_2 = 2(1+y_1)^{-2} \left[-e^{-(1+y_1)} \right]_0^\infty$$
$$= 2(1+y_1)^{-2} I_{(0,1)}(y_1)$$ ✓✓

(b) $E[Y_2|Y_1] = \int_{Y_2} y_2 \cdot f_{2|1}(y_2|y_1) dy_2$

$$f_{2|1}(y_2|y_1) = f_{1,2}(y_1,y_2) / f_1(y_1) = 2y_2 e^{-y_2(1+y_1)} I_{(0<y_1<1)} I_{(0<y_2<+\infty)} / 2(1+y_1)^{-2} I_{(0,1)}(y_1)$$
$$= (1+y_1)^2 y_2 e^{-y_2(1+y_1)} I_{(0<y_2<+\infty)}$$

$$\therefore E[Y_2|Y_1] = \int_0^\infty (1+y_1)^2 y_2^2 e^{-y_2(1+y_1)} dy_2$$
$$= (1+y_1)^2 \int_0^\infty y_2^{\alpha-1} e^{-y_2/(1+y_1)^{-1}} dy_2 \quad (\alpha = 2, \beta = (1+y_1)^{-1}) \rightarrow \sim \text{Gamma}(2, (1+y_1)^{-1})$$
$$= (1+y_1)^2 \cdot 2! (1+y_1)^{-3} = 2(1+y_1)^{-1}$$

$$Var(Y_2|Y_1) = E[Y_2^2|Y_1] - E[Y_2|Y_1]^2$$
$$= 6(1+y_1)^{-2} - 4(1+y_1)^{-2}$$
$$= 2(1+y_1)^{-2}$$ □

4.7. 확률변수 X_1, X_2의 결합확률밀도함수가

$$f(x_1,x_2) = 2e^{-x_1-x_2} I_{(0<x_1<x_2)}$$

일 때, $Y = X_1 + X_2$의 확률밀도함수를 구하여라.

let $Y = X_1 + X_2$ and $Z = X_1 - X_2$

$$\mathcal{X} := \{ (x_1,x_2) : 0 < x_1 < x_2 < +\infty \}$$
$$(Y,Z) := \{ (y,z) : 0 < z < y \}$$

$$u(x) : \begin{cases} Y = X_1 + X_2 \\ Z = X_2 \end{cases}, \quad u^{-1}(Y,z) : \begin{cases} X_1 = (Y+z)/2 \\ X_2 = (Y-z)/2 \end{cases}$$

$$|J_{u^{-1}}| = \left| \det\left(\frac{\partial x}{\partial y}\right) \right| = \begin{vmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{vmatrix} = 1/2$$

(1) $u : \mathcal{X} \mapsto (Y,Z)$ is one-to-one
(2) $\mathcal{X}$ (domain) is an open set in $\mathbb{R}^2$
(3) $|J_{u^{-1}}| \neq 0$

$$f_X(x) = f_{Y,Z}(u^{-1}(x)) |J_{u^{-1}}| = 2e^{-y} \cdot \frac{1}{2} \cdot I_{(0<z<y)}$$

$$f_Y(y) = \int_0^y e^{-y} dz = y e^{-y} I_{(0<y)}$$

[Textbook Solution]

$$u : \begin{cases} Y = X_1 + X_2 \\ Z = X_2 \end{cases}, \quad u^{-1} : \begin{cases} x_1 = y - z \\ x_2 = z \end{cases}, \quad u : 1\text{-}1$$

$$pdf_{Y,Z}(y,z) = 2e^{-x_1-x_2} I_{(0<x_1<x_2)} \left| \det\left(\frac{\partial(x_1,x_2)}{\partial(y,z)}\right) \right|$$

$$\therefore pdf_Y(y) = \int_{-\infty}^{+\infty} pdf_{Y,Z}(y,z)\,dz$$

$$= \int_{y/2}^{y} 2e^{-y}\,dz\; I_{(y>0)}$$

$$= ye^{-y} I_{(0,\infty)}(y)$$

4.8. 확률변수 X의 확률밀도함수가 다음과 같은 경우에 $Y=X^2$의 확률밀도함수를 구하라.

(a) $f(x) = \frac{1}{4} I_{(-2,2)}(x)$ (b) $f(x) = \frac{1}{4} I_{(-1,3)}(x)$

(a) $\mathcal{X} := \{x : -2 \le x \le 2\}$, $\mathcal{Y} := \{y : 0 \le y < 4\}$

$u(x) = x^2$, $u^{-1}(Y) = \pm y^{1/2}$

(1) domain $\mathcal{X}$ is an open set

(2) $u : \mathcal{X} \mapsto \mathcal{Y}$ is 2-to-1

(3) $|J_{u^{-1}}| = \left|\det\left(\frac{\partial x}{\partial y}\right)\right| = \left|\pm\frac{1}{2} y^{-\frac{1}{2}}\right| = \frac{1}{2} y^{-\frac{1}{2}}$

$$f_X(x) = \sum_{x:u(x)=y} f_X(u^{-1}(x))\,|J_{u^{-1}}| = 2\cdot\frac{1}{4} I_{(0,4)}(y)\cdot\frac{1}{2} y^{-\frac{1}{2}}$$
$$= \frac{1}{4} y^{-1/2} I_{(0,4)}(y)$$ ✓✓

(b) $f_X(x) = \frac{1}{4} I_{(-1,3)}(x) = \frac{1}{4} I_{(-1,1)}(x) + \frac{1}{4} I_{(1,3)}(x)$

$$= 2\cdot\frac{1}{4} I_{(0,1)}(y)\cdot\frac{1}{2} y^{-1/2} + \frac{1}{4} I_{(1,3)}(y)\cdot\frac{1}{2} y^{-1/2}$$

$$= \frac{1}{4} y^{-1/2} I_{(0,1)}(y) + \frac{1}{8} y^{-1/2} I_{(1,9)}(y)$$ ✓✓

4.9. 확률변수 X_1, X_2의 결합확률밀도함수가

$$f(x_1,x_2) = \frac{1}{\pi} I_{(x_1^2+x_2^2<1)}$$

일 때 다음에 답하여라.

(a) 극좌표 형식인 $X_1 = R\cos\Theta$, $X_2 = R\sin\Theta$, $R \ge 0$, $0 \le \Theta < 2\pi$로 정의되는 R, Θ의 결합확률밀도함수를 구하여라.

(b) $Y_1 = \sqrt{X_1^2+X_2^2}$, $Y_2 = \dfrac{X_1}{\sqrt{X_1^2+X_2^2}}$ 일 때 Y_1, Y_2의 결합확률밀도함수를 구하여라.

$$u(X) \begin{cases} R = \sqrt{X_1^2+X_2^2} \\ \Theta = \cos^{-1}\left(X_1/\sqrt{X_1^2+X_2^2}\right) = \sin^{-1}\left(X_2/\sqrt{X_1^2+X_2^2}\right) \end{cases}$$

$$u^{-1}(Y) \begin{cases} X_1 = R\cos\Theta \\ X_2 = R\sin\Theta \end{cases} \quad \text{let } Y = (R,\Theta)$$

$\therefore u : X \mapsto Y$ is 1-1. $\quad |J_{u^{-1}}| = \begin{vmatrix} \cos\Theta & -R\sin\Theta \\ \sin\Theta & R\cos\Theta \end{vmatrix} = R\cos^2\Theta + R\sin^2\Theta = R$

$$f_Y(y) = f_X(u^{-1}(x))\,|J_{u^{-1}}| = \frac{1}{\pi} I_{(0<r<1)} I_{(0<\Theta<2\pi)} \cdot R$$ ✓✓

(b) $u(X) \begin{cases} R = \sqrt{X_1^2+X_2^2} \\ \cos\Theta = X_1/\sqrt{X_1^2+X_2^2} \end{cases}$, $u^{-1}(Y) \begin{cases} X_1 = Y_1 Y_2 \\ X_2 = (Y_1^2 - Y_1^2Y_2^2)^{1/2} = Y_1(1-Y_2^2)^{1/2} \end{cases}$

☆

$$f_Y(y) = \sum_{x:u(x)=y} f_X(|u^{-1}(x)|)\cdot|J_{u^{-1}}|$$

$$u^{-1}(Y) \begin{cases} X_1 = Y_1\cdot Y_2 \\ X_2 = (Y_1^2 - Y_1^2Y_2^2)^{1/2} = Y_1(1-Y_2^2)^{1/2} \end{cases}$$

$$|J_{u^{-1}}| = \begin{vmatrix} y_2 & y_1 \\ (1-y_2)^{1/2} & \frac{1}{2}\cdot -2y_2\cdot y_1(1-y_2^2)^{-1/2} \end{vmatrix} = \left|-y_1y_2(1-y_2^2)^{-1/2} - y_1(1-y_2)^{1/2}\right|$$

$$= \left|(1-y_2^2)^{-1/2}\left(-y_1y_2 - y_1(1-y_2)\right)\right|$$
$$= \left|(1-y_2^2)^{-1/2}\left[-y_1\right]\right| = y_1(1-y_2^2)^{-1/2}$$

$$f_Y(y) = \frac{2}{\pi} y_1(1-y_2^2)^{-1/2} I_{(0,1)}(y_1) I_{(-1,1)}(y_2)$$ ✓✓

4.10. 발생률이 λ인 포아송과정 $\{N_t : t \ge 0\}$에서 r번째 현상이 발생할 때까지의 시간을

$$W_r = \min\{t : N_t \ge r\} \quad (r = 1,2,\cdots)$$

이라고 할 때 다음에 답하여라.

(a) $(Y_1, Y_2, \cdots, Y_k)^t$

$= (w_1/w_{k+1}, (w_2-w_1)/w_{k+1}, \cdots, (w_k - w_{k-1})/w_{k+1})'$의 결합확률밀도함수를 구하여라.

(b) $(X,Y)' = (w_2/w_5, w_4/w_5)'$의 결합확률밀도함수를 구하여라.

(a) $W_r \sim \text{Gamma}(r, 1/\lambda)$

$W_r = w_1 \oplus w_2 - w_1 \oplus \cdots \oplus w_r - w_{r-1}$ $\quad w_i - w_{i-1} \overset{iid}{\sim} \text{Exp}(\lambda^{-1})$, $i = 2,\cdots,k$

$W_{k+1} \sim \text{Gamma}(k+1, 1/\lambda)$

$$u \begin{cases} Y_1 = w_1/w_{k+1} \\ Y_2 = (w_2-w_1)/w_{k+1} \\ \vdots \\ Y_k = (w_k - w_{k-1})/w_{k+1} \\ Y_{k+1} = w_1 + \cdots + w_{k+1} \end{cases} \qquad u^{-1} \begin{cases} w_1 = Y_1 Y_{k+1} \\ w_2 = (Y_2+Y_1)Y_{k+1} \\ \vdots \\ w_k = (Y_k + Y_{k-1} + \cdots + Y_2 + Y_1)Y_{k+1} \\ w_{k+1} = Y_{k+1}(1 - Y_1 - \cdots - Y_k) \end{cases}$$

$Y = (Y_1, \cdots, Y_k)^t$ has a Dirichlet distribution.

$a_1 = \cdots = a_k = 1$, $a_{k+1} = k+1$

$Y \sim \text{Dirichlet}(a_1=1, \cdots, a_k=1)$

[Textbook Solution]

$w_1, w_2 - w_1, \cdots, w_k - w_{k-1}, w_{k+1} - w_k \overset{iid}{\sim} \text{Exp}(1/\lambda)$

(a) $(Y_r)_{1\le r\le k} \overset{d}{\equiv} \left(\dfrac{X_r}{X_1 \oplus \cdots \oplus X_{k+1}}\right)_{1\le r\le k+1} \sim \text{Dirichlet}(1,\cdots,1)$

$X_r \overset{iid}{\sim} \text{Gamma}(1, 1/\lambda)$

(b) $(X_1, Y-X)$

$\overset{d}{=} \left(\dfrac{W_2}{W_2 \oplus (W_4-W_2) \oplus (W_5-W_4)}, \dfrac{W_4-W_2}{W_2 \oplus (W_4-W_2) \oplus (W_5-W_4)} \right) \sim \text{Dirichlet}(2,2,1)$

$(X, Y-X)' = (Z_1, Z_2)$ 라고 하면,

$pdf_{Z_1,Z_2}(z_1,z_2) = \dfrac{\Gamma(5)}{\Gamma(2)\Gamma(2)\Gamma(1)} z_1 z_2 I_{(z_1>0, z_2>0, z_1+z_2<1)}$

$pdf_{X,Y}(x,y) = 24 z_1 z_2 I_{(z_1>0, z_2>0, z_1+z_2<1)} \left| \det\left(\dfrac{\partial(z_1,z_2)}{\partial(x,y)} \right) \right|$

$= 24 x(y-x) I_{(0<x<y<1)}$

4.11. 확률변수 X가 $F(r_1, r_2)$ 분포를 따를 때

$$Y = \frac{(r_1/r_2)X}{1+(r_2/r_1)X}$$

는 $Beta(r_1/2, r_2/2)$ 분포를 따르는 것을 보여라.

F 분포 · Beta 분포의 대의적 정의

$F = \dfrac{V_1/r_1}{V_2/r_2}$, $V_1 \sim \chi^2(r_1)$ and $V_2 \sim \chi^2(r_2)$

$Beta(r_1/2, r_2/2) = \dfrac{u_1}{u_1+u_2}$, $u_1 \sim Gamma(\frac{r_1}{2}, \beta)$, $u_2 \sim Gamma(\frac{r_2}{2}, \beta)$

$X \sim F(r_1, r_2)$

$\left(\dfrac{r_1}{r_2}\right) X \overset{d}{=} \left(\dfrac{r_1}{r_2}\right) \cdot \dfrac{V_1/r_1}{V_2/r_2} \overset{d}{=} V_1/V_2$

$1 + \left(\dfrac{r_1}{r_2}\right) X = 1 + \dfrac{r_1}{r_2} \cdot \dfrac{V_1/r_1}{V_2/r_2} = 1 + \dfrac{V_1}{V_2} = \dfrac{V_1+V_2}{V_2}$

$\therefore \left(\dfrac{r_1}{r_2}\right) X \Big/ \left(1 + \dfrac{r_2}{r_1} X\right) = \dfrac{V_1}{V_2} \cdot \dfrac{V_2}{V_1+V_2} = \dfrac{V_1}{V_1+V_2}$

$V_1 \sim \chi^2(r_1) \overset{d}{=} Gamma(\frac{r_1}{2}, 2)$

$V_2 \sim \chi^2(r_2) \overset{d}{=} Gamma(\frac{r_2}{2}, 2)$

$\therefore \dfrac{V_1}{V_1+V_2} \overset{d}{=} \dfrac{Gamma(\frac{r_1}{2}, 2)}{Gamma(\frac{r_1}{2}, 2) + Gamma(\frac{r_2}{2}, 2)} \overset{d}{=} Beta(\frac{r_1}{2}, \frac{r_2}{2})$ ✓✓

4.12. 확률변수 $X_1, X_2, \cdots, X_n$이 서로 독립이고 동일한 주변확률밀도함수

$$f(x) = 3x^2 I_{(0,1)}(x)$$

를 따를 때 다음에 답하여라.

(a) $Y = \min_{1 \le i \le n} X_i$ 의 누적분포함수를 구하고, 그 도함수로서 확률밀도함수를 구하여라.

(b) $Y = \max_{1 \le i \le n} X_i$ 의 누적분포함수를 구하고, 그 도함수로서 확률밀도함수를 구하여라.

(a) $F(x) = \int_0^x 3x^2 dx = [x^3]_0^x = x^3 I_{(0,1)}(x)$

$P(Y \le y) = P(\min X_i \le y)$

$= 1 - P(\min X_i \ge y)$

$= 1 - P(X \ge y)^n$

$= 1 - P(1 - P(X \le y))^n$

$= 1 - (1-y^3)^n$

$\therefore P(Y=y) = \dfrac{\partial}{\partial y} P(Y \le y) = n(1-y^3)^{n-1} \cdot 3y^2$

(b) $P(Y \le y) = P(\max X_i \le y)$

$= P(X \le y)^n$

$= (y^3)^n I_{(0,1)}(x)$

$\therefore P(Y=y) = n(y^3)^{n-1} \cdot 3y^2 I_{(0,1)}(y)$

4.13. 확률밀도함수 $f(x) = 2x I_{(0,1)}(x)$

를 따르는 세 확률변수에 기초한 순서통계량을 $X_{(1)} < X_{(2)} < X_{(3)}$ 이라고 할 때 다음에 답하여라.

(a) $Y_1 = X_{(1)}/X_{(2)}$, $Y_2 = X_{(2)}/X_{(3)}$, $Y_3 = X_{(3)}$ 가 서로 독립임을 보여라.

$\mathcal{X} : \{ (x_{(1)}, x_{(2)}, x_{(3)}) : 0 < x_{(1)} < x_{(2)} < x_{(3)} < 1 \}$

$\mathcal{Y} : \{ (y_1, y_2, y_3) : 0 < y_1 < 1, 0 < y_2 < 1, 0 < y_3 < 1 \}$

$u \begin{cases} y_1 = x_{(1)}/x_{(2)} \\ y_2 = x_{(2)}/x_{(3)} \\ y_3 = x_{(3)} \end{cases}$, $u^{-1} \begin{cases} x_{(1)} = y_1 y_2 y_3 \\ x_{(2)} = y_2 y_3 \\ x_{(3)} = y_3 \end{cases}$

$|J_{u^{-1}}| = \left| \det\left(\dfrac{\partial x}{\partial y} \right) \right| = \begin{vmatrix} y_2 y_3 & y_1 y_3 & y_1 y_2 \\ 0 & y_3 & y_2 \\ 0 & 0 & 1 \end{vmatrix} = y_2 y_3^2$

Let $X = (X_{(1)}, X_{(2)}, X_{(3)}) = 3! f(x_{(1)}) \cdot f(x_{(2)}) \cdot f(x_{(3)})$

$= 6 \cdot 8 x_{(1)} \cdot x_{(2)} \cdot x_{(3)} I_{(0 < x_{(1)} < x_{(2)} < x_{(3)} < 1)}$

$\int_0^1 \int_0^{x_3} \int_0^{x_2} 48 x_1 x_2 x_3 dx_1 dx_2 dx_3$

$= \int_0^1 \int_0^{x_3} \left[24 x_1^2 \right]_0^{x_2} x_2 x_3 dx_2 dx_3$

$= \int_0^1 \int_0^{x_3} 24 x_2^3 x_3 dx_2 dx_3 = \int_0^1 \left[6 x_2^4 \right]_0^{x_3} dx_3 = \int_0^1 6 x_3^5 dx_3 = \left[x_3^6 \right]_0^1 = 1$

$f_Y(y) = f_X(u^{-1}(x)) |J_{u^{-1}}| = 48 y_1 y_2^2 y_3^3 \cdot y_2 y_3^2$

$= 48 y_1 y_2^3 y_3^5 I_{(0,1)}(y_1) \cdot I_{(0,1)}(y_2) \cdot I_{(0,1)}(y_3)$

$= \left[2 y_1 I_{(0,1)}(y_1) \right] \left[4 y_2^3 I_{(0,1)}(y_2) \right] \left[6 y_3^5 I_{(0,1)}(y_3) \right]$

$\therefore Y_1, Y_2$ and Y_3 are independent. ✓✓

(b) 조건부기대값 $E[X_{(2)}|X_{(3)}]$을 구하여라.

$$f_{2,3}(y_2,y_3)=\left[4y_2^3 I_{(0,1)}(y_2)\right]\left[6y_3^5 I_{(0,1)}(y_3)\right]$$
$$f_3(y_3)=\left[6y_3^5 I_{(0,1)}(y_3)\right]$$

$$u\begin{cases}x_{(2)}=y_2\cdot y_3\\ x_{(3)}=y_3\end{cases}\qquad u^{-1}\begin{cases}y_2=x_{(2)}/x_{(3)}\\ y_3=x_{(3)}\end{cases}$$

$$|J_{u^{-1}}|=\left|\det\left(\frac{\partial y}{\partial x}\right)\right|=\begin{vmatrix}x_3^{-1} & -x_2x_3^{-2}\\ 0 & 1\end{vmatrix}=x_3^{-1}$$

$$f_{x_{(2)},x_{(3)}}(x_2,x_3)=f_{y_1,y_2}(u^{-1}(y))=\left[4x_2^3x_3^{-3}\right]\left[6x_3^5\right]\cdot I_{(0<x_2<x_3<1)}\cdot x_3^{-1}$$
$$=24x_2^3x_3\,I_{(0<x_2<x_3<1)}$$

$$f_{2|3}(x_2|x_3)=f_{2,3}(x_2,x_3)\div f_3(x_3)=4x_2^3x_3^{-4}I_{(0<x_2<x_3)}$$

$$E[X_{(2)}|X_{(3)}]=\int_{x_2\in S_{X_{(2)}}} x_2\cdot f_{2|3}(x_2|x_3)\,dx_2=\int_0^{x_3}4x_2^4x_3^{-4}dx_2$$
$$=\left[\frac{4}{5}x_2^5x_3^{-4}\right]_0^{x_3}=\frac{4}{5}x_3$$ ✓

[Text+book Solution]

(b) $E[X_{(2)}|X_{(3)}]=X_{(3)}E[X_{(2)}/X_{(3)}|X_{(3)}]$
$=X_{(3)}E[Y_2]$
$=X_{(3)}\int_0^1 y_2\cdot 4y_2^3dy_2$

4.14. $U_{(1)}<U_{(2)}<\cdots<U_{(n)}$이 균등분포 $U(0,1)$로부터의 랜덤표본 $U_1,U_2,\cdots,U_n$에 기초한 순서통계량일 때 다음에 답하여라.

(a) $Y_1=U_{(1)}/U_{(2)},\cdots,Y_{n-1}=U_{(n-1)}/U_{(n)}$의 결합확률밀도함수를 구하여라.

(b) $Y_r\ (1\le r\le n-1)$의 주변확률밀도함수를 구하여라.

$$u\begin{cases}Y_1=U_{(1)}/U_{(2)}\\ \vdots\\ Y_{n-1}=U_{(n-1)}/U_{(n)}\\ Y_n=U_{(n)}\end{cases},\qquad u^{-1}\begin{cases}U_{(1)}=Y_1Y_2\cdots Y_n\\ \vdots\\ U_{(n-1)}=Y_{n-1}\cdot Y_n\\ U_{(n)}=Y_n\end{cases}$$

$$|J_{u^{-1}}|=\left|\det\left(\frac{\partial u^{-1}}{\partial Y}\right)\right|=\begin{vmatrix}y_2y_3\cdots y_n & y_1y_3\cdots y_n & \cdots & y_1y_2\cdots y_{n-1}\\ 0 & y_3y_4\cdots y_n & \cdots & y_2y_3\cdots y_{n-1}\\ \vdots & \vdots & & \vdots\\ 0 & 0 & \cdots & 1\end{vmatrix}$$
$$=y_2y_3^2\cdots y_n^{n-1}$$

$$f_U(u_{(1)},\cdots,u_{(n)})=n!\,I_{(0<u_{(1)}<\cdots<u_{(n)}<1)}$$

$$f_Y(y)=f_U(u^{-1}(u))|J_{u^{-1}}|=n!\,y_2y_3^2\cdots y_n^{n-1}I_{(0,1)}(y_1)\cdots I_{(0,1)}(y_n)$$

Let $Y'=(Y_1,\cdots,Y_{n-1})$

$$f_{Y'}(y)=\int_0^1 n!\,y_2y_3^2\cdots y_n^{n-1}dy_n=(n-1)!\,y_2y_3^2\cdots y_{n-1}^{n-2}I_{(0,1)}(y_1)\cdots I_{(0,1)}(y_{n-1})$$ ✓✓

(b) $r\,y_r^{r-1}I_{(0,1)}(y_r)$ ✓✓

4.15. $U_{(1)}<U_{(2)}<\cdots<U_{(n)}$이 균등분포 $U(0,1)$로부터의 랜덤표본 $U_1,U_2,\cdots,U_n$에 기초한 순서통계량일 때

$$(U_{(r)}>p)=\left(\sum_{i=1}^n I_{(0,p]}(U_i)\le r-1\right)$$

임을 설명하고,

이를 이용하여 다음 등식이 성립함을 밝혀라.

$$\int_0^p\frac{n!}{(r-1)!(n-r)!}u^{r-1}(1-u)^{n-r}du=\sum_{k=0}^{r-1}\binom{n}{k}p^k(1-p)^{n-k},\quad 0<p<1$$

(solution)

$U_{(r)}>p \iff U_{(n)},\cdots,U_{(r)}>p \iff$ at most, $U_{(1)},\cdots,U_{(r-1)}\le p$
$\iff$ The number of $U_i\ (1\le i\le n)\le p$ is less than or equal to $r-1$
$\iff$ i.e. $\left(\sum_{i=1}^n I_{(0,p]}(U_i)\le r-1\right)$

pdf of $U_{(r)}$: $\frac{n!}{(r-1)!(n-r)!}F(U_{(r)})^{r-1}\cdot f_x(U_{(r)})\cdot[1-F(U_{(r)})]^{n-r}$

$$=\frac{n!}{(r-1)!(n-r)!}u_r^{r-1}\cdot 1\cdot(1-u_r)^{n-r}\sim Beta(r,n-r+1)$$

$$\therefore P(U_{(r)}>p)=\int_p^1\frac{n!}{(r-1)!(n-r)!}u_r^{r-1}\cdot 1\cdot(1-u_r)^{n-r}du_r$$

$$P\left(\sum_{i=1}^n I_{(0,p]}(U_i)\le r-1\right)=\sum_{i=1}^{r-1}\binom{n}{k}p^i(1-p)^{n-k},\quad 0<p<1$$

$$P(I_{(0,p)}(U_i)=1)=p$$

* $X\sim Beta(r,n-r+1)$: $P(X>p)=\sum_{k=0}^{r-1}\binom{n}{k}p^k(1-p)^{n-k}$, $0<p<1$

4.16 $U_{(1)}<U_{(2)}<\cdots<U_{(n)}\ (n\ge 2)$이 균등분포 $U(0,1)$로부터의 랜덤표본 $U_1,U_2,\cdots,U_n$에 기초한 순서통계량일 때, $Y=U_{(n)}-U_{(1)}$의 확률밀도함수를 구하여라.

$$u\begin{cases}Y=U_{(n)}-U_{(1)}\\ Z=U_{(1)}\end{cases},\qquad u^{-1}\begin{cases}U_{(1)}=Z\\ U_{(n)}=Y+Z\end{cases}$$

$|J_{u^{-1}}|=1$ Let $U=(U_{(1)},U_{(n)})$ and $W=(Y,Z)$

$$f_U(u_1,u_2)=\frac{n!}{1!(n-2)!1!}[F(u_1)]^{1-1}[f(u_1)][F(u_n)-F(u_1)]^{n-2}[f(u_n)][1-F(u_n)]^{1-1}$$
$$=n(n-1)(u_n-u_1)^{n-2}\,I_{(0<u_1<u_n<1)}$$

(check pdf validity) $\int_0^1\int_0^{u_n}n(n-1)(u_n-u_1)^{n-2}du_1du_n$

$$=\int_0^1\left[-n(u_n-u_1)^{n-1}\right]_0^{u_n}du_n$$
$$=\int_0^1 n\,u_n^{n-1}du_n=\left[u_n^n\right]_0^1=1$$

$$f_W(y,z)=f_U(u^{-1}(u_n,u_1))|J_{u^{-1}}|=n(n-1)y^{n-2}I_{(0<y<1)}I_{(0<z<1)}I_{(y+z<1)}$$

Y and Z are independent. $f_Y(y)=n(n-1)y^{n-2}\int_0^{1-y}dz\,I_{(0,1)}(y)$

$$=n(n-1)y^{n-2}(1-y)I_{(0,1)}(y)$$

4.17. $X_{(1)} < X_{(2)} < \cdots < X_{(n)}$ $(n \geq 2)$이 지수분포 Exp(1)로부터의 랜덤표본 $X_1, \cdots, X_n$에 기초한 순서통계량일 때, $Y = X_{(n)} - X_{(1)}$의 확률밀도함수를 구하여라.

$$f_X(x) = e^{-x} I_{(0,\infty)}(x), \quad F(x) = (1 - e^{-x}) I_{(0,\infty)}(x)$$

$$u \begin{cases} Y = X_{(n)} - X_{(1)} \\ Z = X_{(1)} \end{cases} \qquad u^{-1} \begin{cases} X_{(1)} = Z \\ X_{(n)} = Y + Z \end{cases}$$

$|J_{u^{-1}}| = 1$ and u is 1-1

Let $w = (x, z)$ and $x = (x_{(1)}, x_{(n)})$

$$f_X(x) = \frac{n!}{1!(n-2)!1!} e^{-x_1} \left(e^{-x_1} - e^{-x_n}\right)^{n-2} e^{-x_n}$$

$$= \frac{n!}{(n-2)!} e^{-(x_1 + x_2)} \left(e^{-x_1} - e^{-x_n}\right)^{n-2}$$

$$f_W(y,z) = f_X(u^{-1}(x)) |J_{u^{-1}}| = n(n-1) e^{-(y+2z)} \left(e^{-z} - e^{-y-z}\right)^{n-2} I_{(0,+\infty)}(y) I_{(0,\infty)}(z)$$

$$= n(n-1) e^{-y} e^{-2z} e^{-z \cdot (n-2)} \left(1 - e^{-y}\right)^{n-2} I_{(0,+\infty)}(y) I_{(0,\infty)}(z)$$

$$= n(n-1) e^{-y} \cdot e^{-zn} \left(1 - e^{-y}\right)^{n-2} I_{(0,+\infty)}(y) I_{(0,+\infty)}(z)$$

$$f_Y(y) = \int_0^\infty n(n-1) e^{-y} (1 - e^{-y})^{n-2} e^{-n \cdot z} dz \, I_{(0,\infty)}(y)$$

$$= \left[-(n-1) e^{-y} (1 - e^{-y})^{n-2} e^{-nz} \right]_0^\infty I_{(0,\infty)}(y)$$

$$= (n-1) e^{-y} (1 - e^{-y})^{n-2} I_{(0,\infty)}(y) \quad \checkmark\checkmark$$

4.18. 균등분포 $U(0,1)$을 따르는 확률변수 U에 대하여

$$X = \beta \left[-\log(1-U)\right]^{1/\alpha}, \quad \alpha > 0, \ \beta > 0$$

라고 할 때 다음에 답하여라.

(a) X의 확률밀도함수가 다음과 같이 주어지는 것을 밝혀라.

$$f(x) = \frac{\alpha}{\beta^\alpha} x^{\alpha-1} \exp\left\{-x^\alpha / \beta^\alpha\right\} I_{(0,+\infty)}(x)$$

이 분포를 와이블 (Weibull) 분포라고 하며 기호로는 $X \sim \text{Weibull}(\alpha, \beta)$로 나타낸다.

(b) X의 확률밀도함수와 누적분포를 각각 $f(x)$, $F(x)$라고 할 때

$$\frac{f(x)}{1 - F(x)} = \frac{\alpha}{\beta^\alpha} x^{\alpha-1}, \quad \alpha > 0$$

임을 밝혀라. 확률변수 X가 수명을 나타낼 때 이러한 함수는 수명이 x인 순간에서의 사망률 (mortality rate)을 나타내는 함수로서 흔히 위험률 (hazard rate) 함수라고 불리운다.

(a) $u = \beta \left[-\log(1-U)\right]^{1/\alpha} = x$ as $U \to 0 : X \to 0$, $U \to 1 : X \to \infty$

$$-\log(1-U) = (x/\beta)^\alpha$$

$$1 - U = \exp\left\{-(x/\beta)^\alpha\right\}$$

$$\therefore U = 1 - \exp\left\{-(x/\beta)^\alpha\right\}$$

$$\therefore u^{-1} = 1 - \exp\left\{-(x/\beta)^\alpha\right\}$$

$$|J_{u^{-1}}| = \left| \alpha x^{\alpha-1} / \beta^\alpha \exp\left\{-(x/\beta)^\alpha\right\} \right|$$

$$f_U(u) = I_{(0,1)}(u) \Rightarrow f_U(u^{-1}(u)) |J_{u^{-1}}| = I_{(0,\infty)}(x) \cdot \alpha x^{\alpha-1} / \beta^\alpha \exp\left\{-\frac{x^\alpha}{\beta^\alpha}\right\}$$

(b) $$F(x) = \int_0^x \alpha x^{\alpha-1} / \beta^\alpha \cdot \exp\left\{-\frac{x^\alpha}{\beta^\alpha}\right\}$$

$$= \left[-\exp\left\{-\frac{x^\alpha}{\beta^\alpha}\right\}\right]_0^x = 1 - \exp\left\{-\frac{x^\alpha}{\beta^\alpha}\right\}$$

$$\therefore 1 - F(x) = \exp\left\{-\frac{x^\alpha}{\beta^\alpha}\right\}$$

$$f(x) / (1 - F(x)) = \alpha x^{\alpha-1} / \beta^\alpha, \quad \alpha > 0 \quad \square$$

4.19. 균등분포 $U(0,1)$을 따르는 확률변수 U에 대하여

$$X = \log\left(\frac{U}{1-U}\right)$$

라고 할 때 X가 $L(0,1)$을 따르는 것을 밝혀라.

$$e^x = \frac{u}{1-u}$$

$$e^x (1-u) = u$$

$$e^x - e^x u = u$$

$$e^x = u(1 + e^x)$$

$$\therefore u = e^x (1 + e^x)^{-1}$$

$$|J_{u^{-1}}| = \left| \det\left|\frac{\partial u}{\partial x}\right| \right| = \left| e^x (1+e^x)^{-1} - e^{2x} (1+e^x)^{-2} \right|$$

$$= \left| (e^x + e^{2x} - e^{2x}) (1+e^x)^{-2} \right|$$

$$= e^x (1+e^x)^{-2}$$

$$f_x(x) = f_u(u^{-1}(u)) = I_{\{0 < e^x (1+e^x)^{-1} < 1\}} \, e^x (1+e^x)^{-2}$$

$0 < u < 1 \Leftrightarrow -\infty < x < +\infty$ $\because$ as $u \to 0$, $\log\left(\frac{u}{1-u}\right) = -\infty$ and as $u \to 1$, $\log\left(\frac{u}{1-u}\right) = +\infty$.

$$\therefore f_x(x) = e^x (1+e^x)^{-2} I_{(-\infty, +\infty)}(x)$$

4.20. 균등분포 $U(0,1)$을 따르는 확률변수 U에 대하여

$$X = \tan((U-1/2)\pi)$$

라고 할 때 X가 Cauchy$(0,1)$을 따르는 것을 밝혀라.

$u : X = \tan((U-1/2)\pi)$ $\quad u^{-1} : \frac{1}{\pi}\tan^{-1}(x)+\frac{1}{2}$

$* \; y = \tan^{-1}(x) \Leftrightarrow \tan(y) = x \Leftrightarrow 1+\frac{\sin^2(y)}{\cos^2(y)} = 1+\tan^2(y)dy = dx$

$$\frac{dy}{dx} = \frac{1}{1+x^2}$$

$\therefore |J_{u^{-1}}| = \frac{1}{\pi}\cdot(1+x^2)^{-1}$

$* \; y = \tan^{-1}(x)$

$dy/dx = 1/(1+x^2)$

range of x : $-\infty < x < +\infty$

$$f_X(x) = f_U(u^{-1}(u)) = I_{(-\infty<x<+\infty)}(x)\cdot\frac{1}{\pi(1+x^2)}$$

[Textbook Answer]

$\textcircled{H} = (U-1/2)\pi \sim U(-\pi/2, \pi/2)$

$X = \tan\textcircled{H}$ $(-\pi/2 < \theta < \pi/2)$: $\theta = \tan^{-1}(x)$ 1-1

$$\left|\frac{dx}{d\theta}\right| = |\sec^2\theta| = 1+x^2$$

$$pdf_X(x) = \frac{1}{\pi} I_{(-\pi/2,\pi/2)}(\theta)\left|\frac{dx}{d\theta}\right|^{-1}$$

$$= \frac{1}{\pi}\frac{1}{1+x^2} I_{(-\infty,+\infty)}(x)$$

4.21 $U_{(1)} < U_{(2)} < \cdots < U_{(n)}$ 이 균등분포 $U(0,1)$로부터의 랜덤표본 $U_1, U_2, \cdots, U_n$에 기초한 순서통계량일 때 다음을 밝혀라.

(a) $(U_{(r)})_{1\le r\le n} \overset{d}{\equiv} (1-U_{(n-r+1)})_{1\le r\le n}$, $(-\log U_{(r)})_{1\le r\le n} \overset{d}{\equiv} \left(\frac{1}{n}Z_1+\cdots+\frac{1}{r}Z_{n-r+1}\right)_{1\le r\le n}$

$Z_r \overset{iid}{\sim} Exp(1)$

(c) $U_{(n+1)} = 1$ 이라고 할 때

$$\left(-\log\left[U_{(r)}/U_{(r+1)}\right]^r\right)_{1\le r\le n} \overset{d}{\equiv} (Z_{n-r+1})_{1\le r\le n},\quad Z_r \overset{iid}{\sim} Exp(1)$$

(d) $U_{(n+1)} = 1$ 이라고 할 때

$$(U_{(r)}/U_{(r+1)})^r \overset{iid}{\sim} U(0,1),\quad r = 1,\cdots,n$$

(Solution)

(a) $$f_{U_{(r)}}(u) = \frac{n!}{(r-1)!(n-r)!}u^{r-1}(1-u)^{n-r}I_{(0,1)}(u)$$

$$f_{U_{(n-r+1)}}(u) = \frac{n!}{(n-r)!(r-1)!}u^{n-r+1}(1-u)^{r-1}I_{(0,1)}(u)$$

$$f_{1-U_{(n-r+1)}}(u) = \frac{n!}{(n-r)!(r-1)!}(1-u)^{n-r+1}(1-1+u)^{r-1}I_{(0,1)}(u)$$

$$= \frac{n!}{(n-r)!(r-1)!}(1-u)^{n-r+1}u^{r-1}I_{(0,1)}(u) \equiv pdf_{U_{(r)}}(u)$$

(b) $$(-\log U_{(r)})_{1\le r\le n} \overset{d}{\equiv} \left(\frac{1}{n}Z_1+\cdots+\frac{1}{n-r+1}Z_r\right)_{1\le r\le n}$$

$$-\log U_{(r)} \overset{d}{\equiv} -\log(1-U_{(n-r+1)})$$
$$\overset{d}{\equiv} -\log(1-(1-e^{-Y_{(n-r+1)}}))$$
$$\overset{d}{\equiv} -\log(e^{-Y_{(n-r+1)}})$$
$$\overset{d}{\equiv} Y_{(n-r+1)}$$
$$\overset{d}{\equiv} \left(\frac{1}{n}Z_1+\cdots+\frac{1}{n-(n-r+1)+1}Z_{n-r+1}\right)$$

$$\overset{d}{\equiv} \left(\frac{1}{n}Z_1+\cdots+\frac{1}{r}Z_{n-r+1}\right)_{1\le r\le n}$$

(c) $$-\log\left[U_{(r)}/U_{(r+1)}\right]^r$$
$$= -r\log(U_{(r)}) - (-r\log(U_{(r+1)}))$$
$$= +r\left(\frac{1}{n}Z_1+\cdots+\frac{1}{r}Z_{n-r+1}\right) - r\left(\frac{1}{n}Z_1+\cdots+\frac{1}{r+1}Z_{n-r}\right)$$
$$= r\cdot\frac{1}{r}Z_{n-r+1}$$
$$= (Z_{n-r+1})_{1\le r\le n},\quad Z_{n-r+1}\sim Exp(1)$$

(d) $$-\log\left[U_{(r)}/U_{(r+1)}\right]^r \overset{d}{\equiv} (Z_{n-r+1})_{1\le r\le n},\quad Z_r\sim Exp(1)$$

Let $X = -\log\left[U_{(r)}/U_{(r+1)}\right]^r$

$$pdf_X(x) = e^{-x}I_{(0,+\infty)}(x)$$

$$e^{-x} = y \Leftrightarrow x = -\log(y),\quad |J_{u^{-1}}| = \left|\frac{dx}{dy}\right| = -\frac{1}{y}$$

$$f_Y(y) = f_X(u^{-1}(x))|J_{u^{-1}}| = e^{-[-\log y]}\cdot\frac{1}{y}I_{(0,1)}(y)$$
$$= y\cdot\frac{1}{y}\cdot I_{(0,1)}(y)$$
$$= I_{(0,1)}(y) : \text{pdf of } U(0,1)$$

$\therefore (U_{(r)}/U_{(r+1)})^r \sim U(0,1)$

4.22. 함수 $f(x,y) = \frac{1}{2\pi}\exp\left[-\frac{1}{2}(x^2+y^2)\right]\left\{1+xy\exp\left[-\frac{1}{2}(x^2+y^2-2)\right]\right\}$

가 확률밀도함수임을 밝히고, 이 함수가 확률변수 X, Y의 결합확률밀도함수일 때, X, Y의 주변분포가 $N(0,1)$임을 밝혀라. 이로부터 주변분포가 일차원의 정규분포이면서도 결합분포는 이변량 정규분포가 아닐 수 있음을 알 수 있다.

(Solution)

* Alternative find the global minima of $xy\exp\left[-\frac{1}{2}(x^2+y^2-2)\right]$

e^t is a convex function $\therefore \frac{d^2}{dt^2}e^t \ge 0$ and $1+\text{minima} \ge 0$

$f(x) = f(0) + f'(0)x + f''(x^*)x^2 \ge x^*$ s.t. $|x^*-x| \le |x-0|$

$e^t = 1+t+e^{t^*}\cdot t^2 \ge 1+t$

(1) $\exp\left\{\frac{1}{2}(x^2+y^2-2)\right\} \ge 1+\frac{1}{2}(x^2+y^2-2) \ge \frac{1}{2}(x^2+y^2) \ge xy$

↳ 여기서 시작하는 것이 중요

$\therefore f(x,y) \ge 0 \quad \forall x, \forall y$

$(x-y)^2 = (x^2-2xy+y^2) \ge 0$

$\therefore x^2+y^2 \ge 2xy \Rightarrow \frac{1}{2}(x^2+y^2) \ge xy$

$xy\exp\left\{\frac{1}{2}(x^2+y^2-2)\right\} \ge x^2\cdot y^2 \ge 0$

$\exp\left\{-\frac{1}{2}(x^2+y^2-2)\right\} \ge 0$ $\therefore$ multiply it on both sides of the inequality.

(2) $f(x,y) = \phi(x)\phi(y) + xy\,\phi(\sqrt{2}x)\phi(\sqrt{2}y)$, $\phi(z) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}$

$$\therefore \int_{-\infty}^{+\infty} f(x,y)\,dx = \phi(y) \quad (\because \int_{-\infty}^{+\infty}\phi(x)dx = 1,\ \int_{-\infty}^{+\infty} x\,\phi(\sqrt{2}x)dx = 0)$$

$$\therefore \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,y)\,dx\,dy = \int_{-\infty}^{+\infty}\phi(y)\,dy = 1$$

Think of $E[X]$

4.23. 확률변수 X_1, X_2, X_3가 서로 독립이고 각각이 $N(0,1)$을 따를 때, 다음과 같이 이들을 구면좌표로 나타내는 R, Θ_1, Θ_2의 결합확률밀도함수를 구하여라.

$$X_1 = R\cos\theta_1 \sin\theta_2 \;,\; X_2 = R\sin\theta_1\sin\theta_2 \;,\; X_3 = R\cos\theta_2$$
$$0 \le R \le +\infty,\; 0 \le \theta_1 \le 2\pi,\; 0 \le \theta_2 \le \pi$$

$$u^{-1}\begin{cases} X_1 = R\cos\theta_1\sin\theta_2 \\ X_2 = R\sin\theta_1\sin\theta_2 \\ X_3 = R\cos\theta_2 \end{cases}, \quad |J_{u^{-1}}| = \begin{vmatrix} \cos\theta_1\sin\theta_2 & -R\sin\theta_1\sin\theta_2 & R\cos\theta_1\cos\theta_2 \\ \sin\theta_1\sin\theta_2 & R\cos\theta_1\sin\theta_2 & R\sin\theta_1\cos\theta_2 \\ \cos\theta_2 & 0 & -R\sin\theta_2 \end{vmatrix}$$

$$= \left| -\cos\theta_2 \begin{vmatrix} -R\sin\theta_1\sin\theta_2 & R\cos\theta_1\cos\theta_2 \\ R\cos\theta_1\sin\theta_2 & R\sin\theta_1\cos\theta_2 \end{vmatrix} + R\sin\theta_2 \begin{vmatrix} \cos\theta_1\sin\theta_2 & -R\sin\theta_1\sin\theta_2 \\ \sin\theta_1\sin\theta_2 & R\cos\theta_1\sin\theta_2 \end{vmatrix} \right|$$

$$= \left| -\cos\theta_2\left(-R^2\sin^2\theta_1\sin\theta_2\cos\theta_2 - R^2\cos^2\theta_1\sin\theta_2\cos\theta_2\right) + R\sin\theta_2\left(R\cos^2\theta_1\sin^2\theta_2 + R\sin^2\theta_1\sin^2\theta_2\right) \right|$$

$$= \left| R^2\sin\theta_2\cos^2\theta_2 + R^2\sin^3\theta_2 \right| = R^2\sin\theta_2$$

$$f_{X_1,X_2,X_3}(x_1,x_2,x_3) = \left(\frac{1}{\sqrt{2\pi}}\right)^3 \exp\left\{-\frac{1}{2}(x_1^2+x_2^2+x_3^2)\right\}$$

$$f_{R,\Theta_1,\Theta_2}(r,\theta_1,\theta_2) = f_{X_1,X_2,X_3}(u^{-1}(x))\,|J_{u^{-1}}| = (2\pi)^{-3/2}\exp\left\{-\frac{1}{2}\left|r^2\cos^2\theta_1\sin^2\theta_2 + r^2\sin^2\theta_1\sin^2\theta_2 + r^2\cos^2\theta_2\right.\right\}\cdot r^2\sin\theta_2$$

$$= (2\pi)^{-3/2}\exp\left\{-\frac{1}{2}r^2\right\}\cdot r^2\sin\theta_2\, I_{[0,\infty)}(r)\, I_{[0,2\pi]}(\theta_1)\, I_{[0,\pi]}(\theta_2) \;\checkmark$$

[Textbook solution]

$$= \frac{2}{\sqrt{2\pi}} r^2 e^{-\frac{1}{2}r^2} I_{[0,\infty)}(r)\,\frac{1}{2\pi} I_{[0,2\pi)}(\theta_1)\,\frac{1}{2}\sin\theta_2\, I_{[0,\pi]}(\theta_2)$$

R, Θ_2, Θ_2는 서로 독립.

$$\begin{cases} R^2 \sim \chi^2(3)\;;\; (\text{pdf})\;\dfrac{1}{\Gamma(3/2)2^{3/2}}(r^2)^{3/2-1}e^{-r^2/2}\,d(r^2) \\ \Theta_2 \sim U[0,2\pi) \\ \cos\Theta_2 \sim U(-1,1)\;;\; (\text{pdf})\;\dfrac{1}{2}I_{[-1,1]}|d\cos\theta_2| \end{cases}$$

4.24. $X = (X_1, X_2, X_3)^t \sim N(\mu, \Sigma)$이고 μ와 Σ가 아래와 같을 때 다음에 답하여라.

$$\mu = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix},\quad \Sigma = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 1 & 2 \end{pmatrix}$$

(a) $Y = (X_1 - X_2 + X_3,\; 2X_1 + X_2 - X_3)'$의 분포를 구하여라.

(b) $X_3 = x_3$가 주어진 조건에서, $(X_1, X_2)'$의 조건부 분포를 구하여라.

(a) $f_X(x) = |\det(2\pi\Sigma)|^{-1/2}\exp\left\{-\frac{1}{2}(x-\mu)^t\Sigma^{-1}(x-\mu)\right\}$

$X_1 - X_2 + X_3 = (1\ -1\ 1)X$

$2X_1 + X_2 - X_3 = (2\ 1\ -1)X$

$$\therefore A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -1 \end{bmatrix},\quad AX = \begin{bmatrix} X_1 - X_2 + X_3 \\ 2X_1 + X_2 - X_3 \end{bmatrix} = Y$$

$X \sim N(\mu, \Sigma)$, $Y = AX \sim N(A\mu, A\Sigma A^t)$

(a) (continued) $Y \sim N(A\mu, A\Sigma A^t)$

$$f_X(x_1,x_2,x_3) = |\det(2\pi\Sigma)|^{-1/2}\exp\left\{-\frac{1}{2}(x-\mu)^t\Sigma^{-1}(x-\mu)\right\}$$

$$f_Y(y_1,y_2) = |\det(2\pi A\Sigma A^t)|^{-1/2}\exp\left\{-\frac{1}{2}(Y-A\mu)^t(A\Sigma A^t)^{-1}(Y-A\mu)\right\}$$

$$A\mu = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \;\checkmark$$

$$A\Sigma A^t = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -2 & 1 \\ 1 & -1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 1 \\ 1 & 2 \end{bmatrix} \;\checkmark$$

$$(A\Sigma A^t)^{-1} = \frac{1}{5\cdot 2 - 1}\begin{bmatrix} 2 & -1 \\ -1 & 5 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 2 & -1 \\ -1 & 5 \end{bmatrix}$$

$$\det(A\Sigma A^t) = 10 - 1 = 9$$

$$\therefore f_Y(y) = (4\pi^2\cdot 9)^{-1/2}\exp\left\{-\frac{1}{2}(Y-A\mu)^t(A\Sigma A^t)^{-1}(Y-A\mu)\right\}$$

p191 $\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \sim N\left(\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}\right)$ } 외우기!

$X_2|X_1 = x_1 \sim N(\mu_2 + \Sigma_{21}\Sigma_{11}^{-1}(x_1 - \mu_1),\; \Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12})$

$2 \Rightarrow (1,2)$, $1 \Rightarrow 3$

$\mu = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ let $\mu_{12} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\mu_3 = (3)$

$\Sigma = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$, $\Sigma_{11} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$, $\Sigma_{12} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\Sigma_{21} = [0,1]$, $\Sigma_{22} = [2]$

$X_{1,2|3} \sim N(\mu_{12} + \Sigma_{12}\Sigma_{22}^{-1}(x_3 - \mu_3),\; \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21})$

(1) $\Sigma_{12}\Sigma_{22}^{-1} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}\cdot\frac{1}{2} = \begin{bmatrix} 0 \\ 1/2 \end{bmatrix}$

$\therefore \Sigma_{12}\Sigma_{22}^{-1}(x_3 - \mu_3) = [0,\ 1/2(x_3-3)]^t$

(2) $\Sigma_{12}\Sigma_{22}^{-1} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}\cdot\frac{1}{2} = \begin{bmatrix} 0 \\ 1/2 \end{bmatrix}$, $\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21} = \begin{bmatrix} 0 \\ 1/2 \end{bmatrix}[0,1] = \begin{bmatrix} 0 & 0 \\ 0 & 1/2 \end{bmatrix}$

$\Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 1/2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 3/2 \end{bmatrix}$

$$\therefore X_{1,2|3} \sim N\left(\begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{1}{2}(x_3 - 3) \end{pmatrix},\; \begin{bmatrix} 1 & -1 \\ -1 & 3/2 \end{bmatrix}\right) \;\checkmark$$

4.25. 〈예 4.4.6〉과 〈예 4.4.7〉에서의 단순선형회귀모형에서 다음이 성립하는 것을 보여라.

$$\frac{\hat{\beta}_0+\hat{\beta}_1 x_1-(\beta_0+\beta_1 x)}{\sqrt{\hat{\sigma}^2\left(\frac{1}{n}+\frac{(x_1-\bar{x})^2}{S_{xx}}\right)}} \sim t(n-2)$$

$t \sim t(n-2)$, $t \overset{d}{\equiv} \frac{Z}{\sqrt{V/(n-2)}}$, $Z \sim N(0,1)$, $V \sim \chi^2(n-2)$

〈예 4.4.6〉

$$\begin{cases} Y_i = \beta_0 + x_{i1}\beta_1 + e_i \\ e_i \sim N(0,\sigma^2) \end{cases}, \quad i=1,\cdots,n$$

$$(X^TX) = \begin{pmatrix} \mathbf{1}^t \\ x_1^t \end{pmatrix}(\mathbf{1}\ x_1) = \begin{pmatrix} \mathbf{1}^t\mathbf{1} & \mathbf{1}^t x_1 \\ x_1^t\mathbf{1} & x_1^t x_1 \end{pmatrix}$$

$$(X^TX) = \begin{pmatrix} n & \sum x_{i1} \\ \sum x_{i1} & \sum x_{i1}^2 \end{pmatrix}$$

$$(X^TX)^{-1} = \frac{1}{n\sum x_{i1}^2 - (\sum x_{i1})^2}\begin{pmatrix} \sum x_{i1}^2 & -\sum x_{i1} \\ -\sum x_{i1} & n \end{pmatrix}$$

$$= \frac{1}{\sum x_{i1}^2 - n\bar{x}_1^2}\begin{pmatrix} \frac{1}{n}\sum x_{i1}^2 & -\bar{x}_1 \\ -\bar{x}_1 & 1 \end{pmatrix}$$

$$= \frac{1}{S_{xx}}\begin{pmatrix} \frac{1}{n}\sum x_{i1}^2 & -\bar{x}_1 \\ -\bar{x}_1 & 1 \end{pmatrix}$$

$$S_{xx} = \sum_{i=1}^n (x_{i1}-\bar{x}_1)^2 = \sum_{i=1}^n (x_{i1}^2 - 2x_{i1}\bar{x}_1 + \bar{x}_1^2)$$
$$= \sum_{i=1}^n x_{i1}^2 - 2\bar{x}_1\sum_{i=1}^n x_{i1} + n\bar{x}_1^2$$
$$= \sum_{i=1}^n x_{i1}^2 - 2\bar{x}_1(n\bar{x}_1) + n\bar{x}_1^2$$
$$= \sum_{i=1}^n x_{i1}^2 - n\bar{x}_1^2$$

$$\hat{\beta} \sim N(\beta, \sigma^2(X^TX)^{-1}) \overset{d}{\equiv} N\left(\beta, \frac{\sigma^2}{S_{xx}}\begin{pmatrix} \frac{1}{n}\sum x_{i1}^2 & -\bar{x}_1 \\ -\bar{x}_1 & 1 \end{pmatrix}\right)$$

$$(\hat{\beta}_0+x_1\hat{\beta}_1) - (\beta_0+x_1\beta_1) \sim N\left(0, \frac{\sigma^2}{S_{xx}}\cdot(1\ x_1)\begin{pmatrix} \frac{1}{n}\sum x_{i1}^2 & -\bar{x}_1 \\ -\bar{x}_1 & 1 \end{pmatrix}\begin{pmatrix} 1^t \\ x_1^t \end{pmatrix}\right)$$

$$(1\ x_1)\begin{pmatrix} \frac{1}{n}\sum x_{i1}^2 & -\bar{x}_1 \\ -\bar{x}_1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ x_1 \end{pmatrix} = \frac{1}{n}\sum x_{i1}^2 - 2x_1\bar{x}_1 + x_1^2 = \frac{1}{n}\sum x_{i1}^2 + (x_1^2 - 2x_1\bar{x}_1 + \bar{x}_1^2) - \bar{x}_1^2$$
$$= \frac{1}{n}\sum x_{i1}^2 - \bar{x}_1^2 + (x_1-\bar{x}_1)^2$$
$$= \frac{1}{n}(\sum x_{i1}^2 - n\bar{x}_1) + (x_1-\bar{x}_1)^2$$
$$= \frac{1}{n}S_{xx} + (x_1-\bar{x}_1)^2$$

$$\therefore\ x\hat{\beta} - x\beta \sim N\left(0, \sigma^2\left(\frac{1}{n}+\frac{(x_1-\bar{x}_1)^2}{S_{xx}}\right)\right)$$

$$\Rightarrow (x\hat{\beta}-x\beta)\Big/\left(\sigma^2\left(\frac{1}{n}+\frac{(x_1-\bar{x}_1)^2}{S_{xx}}\right)\right)^{1/2} \sim N(0,1)$$

$$(n-2)\hat{\sigma}^2/\sigma^2 \sim \chi^2(n-2)$$

$$\therefore \frac{x\hat{\beta}-x\beta}{\sqrt{\hat{\sigma}^2\left(\frac{1}{n}+\frac{(x_1-\bar{x}_1)^2}{S_{xx}}\right)}} = \frac{(x\hat{\beta}-x\beta)\Big/\left(\sigma^2\left(\frac{1}{n}+\frac{(x_1-\bar{x}_1)^2}{S_{xx}}\right)\right)^{1/2}}{\sqrt{\frac{(n-2)\hat{\sigma}^2}{\sigma^2}\Big/(n-2)}} \overset{d}{\equiv} \frac{Z}{\sqrt{V/(n-2)}} \sim t(n-2)\ \square.$$

[Textbook Solution]

$$(1, x_1)\frac{\sigma^2}{S_{xx}}\begin{pmatrix} \sum x_{i1}^2/n & -\bar{x}_1 \\ -\bar{x}_1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ x_1 \end{pmatrix} = \left(\frac{1}{n}+\frac{(x_1-\bar{x}_1)^2}{S_{xx}}\right)\sigma^2 \quad \hat{\beta}_0+\hat{\beta}_1 x_1 = (1,x_1)\hat{\beta},\ \hat{\beta}=(\hat{\beta}_0,\hat{\beta}_1)^t$$

$(1, x_1)\beta = \beta_0 + \beta_1 x_1$, $\beta = (\beta_0, \beta_1)^t$

$$\therefore \begin{cases} \hat{\beta}_0+\hat{\beta}_1 x_1 \sim N\left(\beta_0+\beta_1 x_1, \left(\frac{1}{n}+\frac{(x_1-\bar{x})^2}{S_{xx}}\right)\sigma^2\right) \\ (n-2)\hat{\sigma}^2/\sigma^2 \sim \chi^2(n-2) \end{cases}$$

$(\hat{\beta}_0+\hat{\beta}_1 x_1$ 와 $(n-2)\hat{\sigma}^2/\sigma^2$ 은 서로 독립.

4.26 $(X_1, Y_1)^t, \cdots, (X_n, Y_n)^t$ $(n>2)$ 이 이변량 정규분포

$$N_2\left(\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}\right), \quad \sigma_1^2>0,\ \sigma_2^2>0,\ -1<\rho<1$$

에서의 랜덤표본이고, r이 표본상관계수 즉

$$r = \frac{\sum_1^n (X_i-\bar{X})(Y_i-\bar{Y})}{\sqrt{\sum_1^n (X_i-\bar{X})^2 \sum_{i=1}^n (Y_i-\bar{Y})^2}}$$

이라고 하자. 또한,

$$Z_i = (X_i-\mu_1)/\sigma_1$$
$$W_i = \left\{(Y_i-\mu_2)/\sigma_2 - \rho(X_i-\mu_1)/\sigma_1\right\}/\sqrt{1-\rho^2} \quad (i=1,\cdots,n)$$

이라고 할 때, 다음에 답하여라.

(a) $\frac{r}{\sqrt{1-r^2}} = \frac{S_{zz}\rho/\sqrt{1-\rho^2} + S_{zw}}{\sqrt{S_{zz}S_{ww} - S_{zw}^2}}$ 임을 보여라. 여기에서

$$S_{zz} = \sum_1^n (Z_i-\bar{Z})^2,\ S_{ww} = \sum_1^n (W_i-\bar{W})^2,\ S_{zw} = \sum_1^n (Z_i-\bar{Z})(W_i-\bar{W})$$

let $V_i = (Y_i-\mu_2)/\sigma_2$.

Then $(X_i-\bar{X})/\sigma_1 = (Z_i-\bar{Z})$, $(Y_i-\bar{Y})/\sigma_2 = (V_i-\bar{V})$

$$V_i - \bar{V} = \sqrt{1-\rho^2}(W_i-\bar{W}) + \rho(Z_i-\bar{Z})$$

$$r = \frac{\sum(Z_i-\bar{Z})(V_i-\bar{V})}{\sqrt{\sum(Z_i-\bar{Z})^2\sum(V_i-\bar{V})^2}} = \frac{S_{zv}}{\sqrt{S_{zz}S_{vv}}} \quad (\because r \text{ is independent of location \& scale})$$

$$\frac{r}{\sqrt{1-r^2}} = \frac{S_{zv}}{\sqrt{S_{zz}S_{vv}}}\Big/\sqrt{1-\frac{S_{zv}^2}{S_{zz}S_{vv}}} = \frac{S_{zv}}{\sqrt{S_{zz}S_{vv}-S_{zv}^2}}$$

$$S_{zv} = \sum(Z_i-\bar{Z})(V_i-\bar{V}) = \sum(Z_i-\bar{Z})\left\{\sqrt{1-\rho^2}(W_i-\bar{W}) + \rho(Z_i-\bar{Z})\right\}$$
$$= \sqrt{1-\rho^2}S_{zw} + \rho S_{zz}$$

$$S_{vv} = \sum(V_i-\bar{V})(V_i-\bar{V}) = \sum\left\{\sqrt{1-\rho^2}(W_i-\bar{W}) + \rho(Z_i-\bar{Z})\right\}\left\{\sqrt{1-\rho^2}(W_i-\bar{W}) + \rho(Z_i-\bar{Z})\right\}$$
$$= (1-\rho^2)S_{ww} + 2\sqrt{1-\rho^2}\,\rho S_{zw} + \rho^2 S_{zz}$$

$$S_{zz}S_{vv} - S_{zv}^2 = (1-\rho^2)S_{zz}S_{ww} + 2\sqrt{1-\rho^2}\cdot\rho S_{zz}\cdot S_{zw} + \rho^2 S_{zz}^2$$
$$-\left[(1-\rho^2)S_{zw}^2 + 2\sqrt{1-\rho^2}\cdot\rho S_{zz}\cdot S_{zw} + \rho^2 S_{zz}^2\right]$$
$$= (1-\rho^2)\left\{S_{zz}S_{ww} - S_{zw}^2\right\}$$

$$\therefore \frac{r}{\sqrt{1-r^2}} = \frac{S_{zv}}{\sqrt{S_{zz}S_{vv}-S_{zv}^2}} = \frac{\sqrt{1-\rho^2}S_{zw} + \rho S_{zz}}{\sqrt{(1-\rho^2)\left\{S_{zz}S_{ww}-S_{zw}^2\right\}}} = \frac{S_{zw} + \rho/\sqrt{1-\rho^2}\cdot S_{zz}}{\sqrt{S_{zz}S_{ww}-S_{zw}^2}}$$

[Textbook]

(a) $X_i - \bar{X} = \sigma_1(Z_i - \bar{Z})$

(b) $Y_i - \bar{Y} = \sigma_2\left\{\rho(Z_i-\bar{Z}) + \sqrt{1-\rho^2}(W_i-\bar{W})\right\}$ $(i=1,\cdots,n)$

임을 이용하여 계산하면 된다.

(b) $T = (Z_1-\bar{Z})/\sqrt{S_{ZZ}}, \cdots, (Z_n-\bar{Z})/\sqrt{S_{ZZ}})'$ 라고 할 때, 다음을 밝혀라.

$$\{S_{WW} - S_{ZW}^2/S_{ZZ} = W'\{I - \mathbb{1}(\mathbb{1}^t\mathbb{1})^{-1}\mathbb{1}^t - T(T^tT)^{-1}T^t\}W$$

Let $Z=(Z_1,\cdots,Z_n)^t$, $\Pi_{\mathbb{1}} = \mathbb{1}(\mathbb{1}^t\mathbb{1})^{-1}\mathbb{1}^t$

$\mathbb{1}\bar{Z} = (\bar{Z},\cdots,\bar{Z})^t = \Pi_{\mathbb{1}}Z$

$S_{ZZ} = \sum_1^n (Z_i-\bar{Z})^2 = (Z-\mathbb{1}\bar{Z})^t(Z-\mathbb{1}\bar{Z}) = (Z-\Pi_{\mathbb{1}}Z)^t(Z-\Pi_{\mathbb{1}}Z)$

$= Z^t(I-\Pi_{\mathbb{1}})^t(I-\Pi_{\mathbb{1}})Z$

$= Z^t(I-\Pi_{\mathbb{1}})Z$ ($\because I-\Pi_{\mathbb{1}}$ is symmetric and idempotent)

$T = (I-\Pi_{\mathbb{1}})Z / (Z^t(I-\Pi_{\mathbb{1}})Z)^{1/2}$

notice that $\bar{Z}=0 \Leftrightarrow \Pi_{\mathbb{1}}Z = 0 \quad \therefore T = Z/(Z^tZ)^{1/2}$

(a) $X_i-\bar{X} = \sigma_1(Z_i-\bar{Z})$

(b) $Y_i-\bar{Y} = \sigma_2\{\rho(Z_i-\bar{Z}) + \sqrt{1-\rho^2}(W_i-\bar{W})\}$

$S_{WW} = \sum_1^n (W_i-\bar{W})^2 = (W-\Pi_{\mathbb{1}}W)^t(W-\Pi_{\mathbb{1}}W) = W^t(I-\Pi_{\mathbb{1}})W$

$S_{ZW} = (Z-\Pi_{\mathbb{1}}Z)^t(W-\Pi_{\mathbb{1}}W) = Z^t(I-\Pi_{\mathbb{1}})^t(I-\Pi_{\mathbb{1}})W = Z^t(I-\Pi_{\mathbb{1}})^tW$

$S_{ZZ} = (Z-\Pi_{\mathbb{1}}Z)^t(Z-\Pi_{\mathbb{1}}Z) = Z^t(I-\Pi_{\mathbb{1}})^t(I-\Pi_{\mathbb{1}})Z = Z^t(I-\Pi_{\mathbb{1}})Z$

$S_{ZW}^2/S_{ZZ} = (Z^t(I-\Pi_{\mathbb{1}})^tW)(Z^t(I-\Pi_{\mathbb{1}})^tW)$

$T = (Z-\Pi_{\mathbb{1}}Z)\cdot(S_{ZZ})^{-1/2}$

$T^tT = (S_{ZZ})^{-1}\cdot Z^t(I-\Pi_{\mathbb{1}})Z = (S_{ZZ})^{-1}S_{ZZ} = I_n$

$(T^tT)^{-1} = I_n$

$T(T^tT)^{-1}T^t = TT^t$

$W^tTT^tW = (\sum_1^n (W_i)(Z_i-\bar{Z})/\sqrt{S_{ZZ}})^2 = \frac{1}{S_{ZZ}}(\sum_1^n (W_i)(Z_i-\bar{Z}))^2$

$= \frac{1}{S_{ZZ}}(\sum_1^n (W_i-\bar{W})(Z_i-\bar{Z}))^2 = S_{WZ}^2/S_{ZZ}$

$\therefore W^tT(T^tT)^{-1}T^tW = S_{WZ}^2/S_{ZZ}$

$\therefore S_{WW} - S_{WZ}^2/S_{ZZ} = W^t(I-\Pi_{\mathbb{1}})W - W^t\Pi_TW$

$= W^t(I-\Pi_{\mathbb{1}}-\Pi_T)W$

(c) $(Z_1,\cdots,Z_n)^t$ 과 $(W_1,\cdots,W_n)$이 서로 독립임을 밝히고, 이를 이용하여 다음을 밝혀라.

(i) $S_{WW} - S_{ZW}^2/S_{ZZ} \sim \chi^2(n-2)$ 이고, $(Z_1,\cdots,Z_n)^t$ 과는 독립이다.

(ii) $S_{ZW}/\sqrt{S_{ZZ}} \sim N(0,1)$ 이고 $(Z_1,\cdots,Z_n)^t$ 과는 독립이다.

(iii) $S_{WW} - S_{ZW}^2/S_{ZZ}$, $S_{ZW}/\sqrt{S_{ZZ}}$, S_{ZZ}는 서로 독립이다.

(i) $ZW = \frac{1}{\sqrt{1-\rho^2}}\left\{\frac{(X_i-\mu_1)(Y_i-\mu_2)}{\sigma_1\sigma_2} - \rho\frac{(X_i-\mu_1)^2}{\sigma_1^2}\right\}$

$E[ZW] = \frac{1}{\sqrt{1-\rho^2}}\left\{\rho - \rho\frac{\sigma_1^2}{\sigma_1^2}\right\} = \frac{1}{\sqrt{1-\rho^2}}\{\rho-\rho\} = 0$

$E[Z] = E[W] = 0$ X INCORRECT.

$\therefore E[ZW] = E[Z]E[W]$

$\therefore Z$ and W are independent.

$A = \begin{pmatrix} -\rho/\sigma_1 & 1/\sigma_2 \end{pmatrix}\cdot\frac{1}{\sqrt{1-\rho^2}}$

$A\left(\begin{pmatrix}X\\Y\end{pmatrix} - \begin{pmatrix}\mu_1\\\mu_2\end{pmatrix}\right) \sim N\left(A\begin{pmatrix}0\\0\end{pmatrix}, \frac{1}{1-\rho^2}\cdot \underbrace{\begin{pmatrix} -\rho/\sigma_1 & 1/\sigma_2 \end{pmatrix}\begin{pmatrix}\sigma_1^2 & \rho\sigma_1\sigma_2\\ \rho\sigma_1\sigma_2 & \sigma_2^2\end{pmatrix}\begin{pmatrix}-\rho/\sigma_1\\1/\sigma_2\end{pmatrix}}_{*}\right)$

$* \quad -\rho/\sigma_1\cdot\sigma_1^2 = -\rho\sigma_1 \qquad -\rho/\sigma_1\cdot\rho\sigma_1\sigma_2 = -\rho^2\sigma_2$

$1/\sigma_2\cdot\rho\sigma_1\sigma_2 = \rho\sigma_1 \qquad 1/\sigma_2\cdot\sigma_2^2 = \sigma_2$

$\begin{pmatrix} 0 & -\rho^2\sigma_2+\sigma_2\end{pmatrix}\begin{pmatrix}-\rho/\sigma_1\\1/\sigma_2\end{pmatrix} = -\rho^2+1 = 1-\rho^2 = *$

$\therefore A\left(\begin{pmatrix}X\\Y\end{pmatrix} - \begin{pmatrix}\mu_1\\\mu_2\end{pmatrix}\right) \sim N(0,1)$

$\therefore W^t(I-\Pi_{\mathbb{1}}-\Pi_T)W \sim \chi^2(r)$, $r = \text{rank}(I-\Pi_{\mathbb{1}}-\Pi_T)$

$= \text{trace}(I-\Pi_{\mathbb{1}}-\Pi_T)$

$= \text{tr}(I) - \text{tr}(\Pi_{\mathbb{1}}) - \text{tr}(\Pi_T)$

$= n-1-1 = n-2$

$\therefore W^t(I-\Pi_{\mathbb{1}}-\Pi_T)W \sim \chi^2(n-2)$

(ii) $S_{ZW}/\sqrt{S_{ZZ}}$

$S_{ZW} = \sum_1^n (Z_i-\bar{Z})(W_i-\bar{W}) = (Z-\Pi_{\mathbb{1}}Z)^t(W-\Pi_{\mathbb{1}}W)$

$S_{ZW}/\sqrt{S_{ZZ}} \sim N(0,1) \Leftrightarrow S_{ZW}^2/S_{ZZ} \sim \chi^2(1)$

$S_{ZW}^2 = (Z^tW)^t(Z^tW) = W^tZZ^tW$

$S_{ZW}^2/S_{ZZ} = W^tZ(Z^tZ)^{-1}Z^tW \sim \chi^2(1)$ $\because W\sim N(0,1)$ and $\text{trace}(Z(Z^tZ)^{-1}Z^t) = 1$

$S_{ZW}^2/S_{ZZ} \in C_W$ (column space of W)

and

$Z = (Z_1,\cdots,Z_n)^t \in C_Z$ (column space of Z)

Z and W are independent.

Hence, naturally, any vectors that are in C_W are independent to the vectors in C_Z.

(iii) $S_{WW} - S_{ZW}^2/S_{ZZ}$, $S_{ZW}/\sqrt{S_{ZZ}} \in C_W$ and $S_{ZZ} \in C_Z$

$\therefore S_{WW} - S_{ZW}^2/S_{ZZ}$, $S_{ZW}/\sqrt{S_{ZZ}}$ are independent to C_Z

INCORRECT

X $E[(S_{WW} - S_{ZW}^2/S_{ZZ})\cdot S_{ZW}/\sqrt{S_{ZZ}}] = E[S_{WW} - S_{ZW}^2/S_{ZZ}]E[S_{ZW}/\sqrt{S_{ZZ}}] = 0$

(d) 다음이 성립함을 밝혀라.

$$\frac{r}{\sqrt{1-r^2}} \overset{d}{\equiv} \frac{\sqrt{V_1}\rho/\sqrt{1-\rho^2} + U}{\sqrt{V_2}}$$

$V_1 \sim \chi^2(n-1)$, $V_2 \sim \chi^2(n-2)$, $U \sim N(0,1)$ 이고 V_1, V_2, U는 서로 독립.

$$\frac{r}{\sqrt{1-r^2}} = \frac{S_{ZZ}\rho/\sqrt{1-\rho^2} + S_{ZW}}{\sqrt{S_{ZZ}S_{WW} - S_{ZW}^2}} = \frac{S_{ZZ}\rho/\sqrt{1-\rho^2} + S_{ZW}}{\sqrt{S_{ZZ}}\sqrt{S_{WW} - S_{ZW}^2/S_{ZZ}}}$$

$$= \frac{\sqrt{S_{ZZ}}\rho/\sqrt{1-\rho^2} + S_{ZW}/\sqrt{S_{ZZ}}}{\sqrt{S_{WW} - S_{ZW}^2/S_{ZZ}}} \overset{d}{\equiv} \frac{\sqrt{V_1}\rho/\sqrt{1-\rho^2} + U}{\sqrt{V_2}}$$

V_1, V_2 and U are independent from (c) (i), (ii), (iii)

(b) $S_{WW} = W^t(I - \mathbb{1}(\mathbb{1}^t\mathbb{1})^{-1}\mathbb{1}^t)W$,
$S_{ZW}/S_{ZZ} = T^tW$, $T^tT = 1$

(c) 예 4.4.3과 다변량 정규분포의 성질 (b)로부터

$$X_i \sim N(\mu_1, \sigma_1^2),\ Y_i - \mu_2 - \rho\sigma_2(X_i - \mu)/\sigma_1 \sim N(0, \sigma_2^2(1-\rho^2))$$

$$X_i \perp (Y_i - \mu_2 - \rho\sigma_2(X_i - \mu_1)/\sigma_1)$$

따라서 $(Z_1, \cdots, Z_n)'$과 $(W_1, \cdots, W_n)^t$이 서로 독립이다.

이로부터 (i), (ii), (iii)은 $Z = (Z_1, \cdots, Z_n)^t$가 주어진 조건부분포를 생각하여 다변량 정규분포의 성질 (b)과 (2)를 차례로 적용하면 밝힐 수 있다.

(d) 이는 (a)의 표현에 (c)를 적용하면 명백하다.