7.1. 평균이 μ이고 표준편차가 $\sigma=5$인 정규모집단에서 크기 $n=25$인 랜덤표본을 추출하여 다음의 가설을 검정하려고 한다.

$$H_0: \mu \le 60 \quad vs. \quad H_1: \mu > 60$$

(a) 표본평균 $\bar{x}$의 큰 값에 대하여 귀무가설 H_0을 기각하는 기각역 "$\bar{x} > c$"를 이용하는 검정의 크기가 5% 이도록 c의 값을 정하여라.

(b) $\mu=65$일 때 (a)의 검정의 검정력 $\gamma(65)$를 구하여라.

(c) (a)의 검정의 검정력함수 $\gamma(\mu)$의 그래프를 그려보아라.

(Solution)

(a) $X \sim N(\mu, \sigma^2)$, $\bar{X} \sim N(\mu, \sigma^2/n)$

$P[\bar{x} > c] = 0.05$ under H_0

$$P\left[\frac{\bar{x}-\mu}{\sigma/\sqrt{n}} > \frac{c-\mu}{\sigma/\sqrt{n}}\right] = P\left(z > \frac{c-\mu}{\sigma/\sqrt{n}}\right) = 0.05, \quad z \sim N(0,1)$$

$\therefore 1-\Phi\left(\frac{c-60}{5/\sqrt{25}}\right) = 0.05$, $\frac{c-60}{5/5} = \Phi^{-1}(0.05) = 1.645$

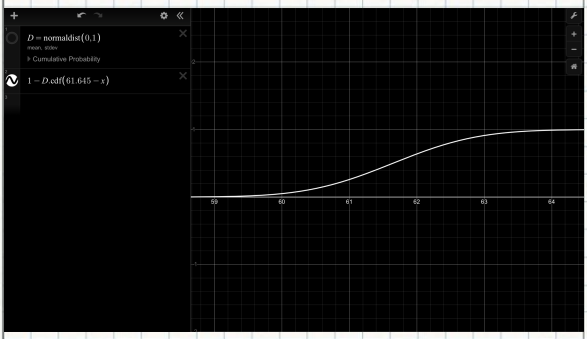
$\therefore c = 61.645$ ✓✓

* $H_0: \mu \le 60$ 이 사실일 때, 기각할 확률이 최대가 되는 것은 $\mu=60$ 이므로

(b) 검정력 : H_1이 사실일 때 H_1을 선택할 확률 $\Leftrightarrow$ H_0을 기각할 확률

$$P_{\mu=65}[\bar{x} > 61.645) = P(z > -3.355) = P(z < 3.355)$$
$$= 1 - 0.000395$$
$$= 0.9996605$$ ✓✓

(c) $P_\mu\left[\frac{\bar{x}-\mu}{\sigma/\sqrt{n}} \ge \frac{61.645-\mu}{\sigma/\sqrt{n}}\right] = 1-\Phi(61.645-\mu)$



7.2. 연습문제 (7.1)에서 유의수준이 5%로 주어지고 $\mu=62$일 때 검정력을 0.9 즉 제 2종 오류를 범할 확률을 0.1로 하려고 할 때 최소한의 표본크기를 구하여라.

$$P_{\mu_0}[\bar{X} \ge C) \le 0.05$$

$$P\left(\frac{\bar{X}-\mu_0}{\sigma/\sqrt{n}} \le \frac{C-\mu_0}{\sigma/\sqrt{n}}\right) = 0.05 \Leftrightarrow P\left(z \le \frac{C-60}{\sigma/\sqrt{n}}\right) = 0.05$$

$$\therefore \frac{C-60}{\sigma/\sqrt{n}} = 1.645 \Rightarrow C = 60 + 1.645 \cdot \frac{5}{\sqrt{n}}$$

$$P_{\mu_1}[\bar{X} \ge C) \ge 0.90 \Leftrightarrow P_{\mu_1}\left[\frac{\bar{X}-\mu_1}{\sigma/\sqrt{n}} \ge \frac{C-\mu_1}{\sigma/\sqrt{n}}\right] \ge 0.90$$

$$\Leftrightarrow P\left(z \ge -\frac{2}{\sigma/\sqrt{n}} + 1.645\right) \ge 0.90$$

$$\Leftrightarrow P\left(z \le \frac{2}{\sigma/\sqrt{n}} - 1.645\right) \ge 0.90$$

$$\Leftrightarrow \frac{2}{\sigma/\sqrt{n}} - 1.645 \ge +1.285$$

$$\Leftrightarrow \sqrt{n} \ge (1.285 + 1.645) \cdot \frac{5}{2}$$
$$= 2.930 \cdot 2.5$$
$$= 7.325$$

$$n \ge 53.655$$

$\therefore n \ge 54$

7.3. 포아송분포 Poisson(μ) 모형에서 크기 $n=10$인 랜덤표본 $X_1, \cdots, X_{10}$을 추출하여 다음의 가설을 검정하려고 한다.

$$H_0: \mu = 0.5 \quad vs. \quad H_1: \mu > 0.5$$

(a) 귀무가설 H_0을 기각하는 기각역 "$X_1 + \cdots + X_{10} \ge C$"를 이용하는 검정의 유의수준이 5%이기 위한 가장 작은 정수 C의 값을 구하여라.

(b) (a)의 결과를 이용하여 귀무가설 $H_0: \mu = 0.5$가 사실일 때 이를 기각하는 확률이 5%인 랜덤화 검정을 구하여라.

(c) (b)의 검정의 검정력함수 $\gamma(\mu)$의 그래프를 그려라.

(Solution)

(a) $X_1 + \cdots + X_{10} \sim$ Poisson(10μ), under H_0, $\sum_{i=1}^{10} X_i \sim$ Poisson(5)

$P(\sum X_i \le 10) = 0.9863$ $\quad P(\sum X_i \ge 11) = 0.01371$

$P(\sum X_i \le 9) = 0.9682$ $\quad P(\sum X_i \ge 10) = 0.0318$

$P(\sum X_i \le 8) = 0.9319$ $\quad P(\sum X_i \ge 9) = 0.0681$

$P(\sum X_i \le 7) = 0.8666$ $\quad P(\sum X_i \ge 8) = 0.1333$

$\therefore C = 10$ $\because P(\sum X_i \ge 10) = 1 - P(\sum X_i \le 9) = 1 - 0.9682 < 0.05$

(b)
- if $\sum X_i \ge 10$: Reject H_0
- if $\sum X_i = 9$: Reject H_0 with probability γ
- if $\sum X_i \le 8$: Accept H_0

$P(\sum X_i \ge 10) + \gamma P(\sum X_i = 9) = 0.05 \Leftrightarrow 0.0318 + \gamma(0.9682 - 0.9319) = 0.05$

$\therefore \gamma = 0.501$

(c) Ask ChatGPT

7.4. 서로 독립인 베르누이 Bernoulli(p) 시행을 7회 한 결과 $X_1, \cdots, X_7$을 이용하여 다음의 가설을 검정하려고 한다.

$$H_0 : p = 0.5 \quad vs \quad H_1 : p > 0.5$$

(a) 귀무가설 H_0을 기각하는 기각역 "$X_1 + \cdots + X_7 \geq c$"을 이용하는 검정의 유의수준이 10% 이기 위한 가장 작은 정수 c의 값을 구하여라.

(b) (a)의 결과를 이용하여 귀무가설 $H_0 : p = 0.5$가 사실일 때 이를 기각하는 확률이 10%인 랜덤화 검정을 구하여라.

(c) (b)의 검정의 검정력함수 $\gamma(p)$의 그래프를 그려라.

(a) $\sum X_i \sim Bin(n, p)$ $\therefore \sum_{i=1}^{7} X_i \sim Bin(7, p)$

$$P(X = x) = \binom{7}{x} p^x (1-p)^{7-x}$$

$$P_{p_0}(X = x) = \binom{7}{x} 0.5^7 = \binom{7}{x} \frac{1}{128}$$

$X = x$	0	1	2	3	4	5	6	7
$P(X = x)$	1/128	7/128	21/128	35/128	35/128	21/128	7/128	1/128
$P(X \leq x)$	1/128	8/128	29/128	64/128	99/128	120/128	127/128	1

$P(X \leq 5) = 0.9375$ $\quad P(X \leq 4) = 0.773$

$P(X \leq 6) = 0.992$

$\therefore c = 6$ $\because P(X \geq 7) = 1 - P(X \leq 6) = 0.008$

$P(X \geq 6) = 1 - P(X \leq 5) = 0.0625$

$P(X \geq 5) = 1 - P(X \leq 4) = 0.227$

(b)
$$\begin{cases} \text{if } \sum X_i \geq 6 & : \text{reject } H_0 \\ \text{if } \sum X_i = 5 & : \text{reject } H_0 \text{ with probability } \gamma \\ \text{if } \sum X_i \leq 4 & : \text{Accept } H_0 \end{cases}$$

$P(\sum X_i \geq 6) + \gamma P(\sum X_i = 5) = 0.10$

$\therefore 0.0625 + \gamma \cdot 0.164 = 0.10$

$\therefore \gamma = \frac{1}{0.164}(0.10 - 0.0625) = 0.228$ ✓✓

(c) Ask ChatGPT

7.5. 일반적으로 검정통계량의 주어진 관측값보다 귀무가설 H_0에 더 반대되는 결과를 관측할 확률을 유의확률(significance probability) 이라고 하며, 이는 그 관측결과에 의해 귀무가설을 기각할 수 있는 최소의 유의수준이다. 연습문제 [7.4]에서 관측결과가

$$X_1 + \cdots + X_7 = 6$$

인 경우에 이 관측결과에 의한 유의확률을 구하여라.

$$P(\sum X_i \geq 6) = \binom{7}{6} p^6 (1-p)^1 + \binom{7}{7} p^7 (1-p)^0 = 7 \cdot 2^{-7} + 2^{-7} = 8 \cdot 2^{-7}$$
$$= 2^{-4} = 1/16$$ ✓✓

Q) 관측값보다 H_0에 더 반대되는 결과를 관측할 확률은 $P(\sum X_i \geq 7)$이 아닌가?

7.6. 정규분포 $N(\mu, \sigma^2)$, $-\infty < \mu < +\infty$, $0 < \sigma^2 < +\infty$ 의 평균에 대한 가설

$$H_0 : \mu = \mu_0 \quad vs. \quad H_1 : \mu \neq \mu_0 \quad (\mu_0 \text{ is given})$$

의 최대가능도비 검정통계량으로서

$$t = \frac{\bar{x} - \mu_0}{s/\sqrt{n}}$$

을 사용하여 양쪽 검정을 하는 것을 〈예 7.2.1〉에서 알고 있다.

(a) 이 t 검정통계량에 대하여 다음과 같은 대의적 정의가 성립하는 것을 밝혀라.[13]

$$t \overset{d}{=} \frac{Z + \delta}{\sqrt{V/(n-1)}}, \quad \delta = \frac{\mu - \mu_0}{\sigma/\sqrt{n}}, \quad Z \sim N(0,1), \quad V \sim \chi^2(n-1)$$
$$Z \text{ and } V \text{ are independent.}$$

(b) 이 양쪽 검정의 검정력함수는 $|\delta| = \left|\frac{\mu - \mu_0}{\sigma/\sqrt{n}}\right|$ 의 함수로서, 즉

$$\gamma(|\delta|) = P_{\mu,\sigma^2}\{|t| \geq t_{\alpha/2}(n-1)\}$$

이고 이 함수가 $|\delta|$의 증가함수임을 밝혀라.

13) 이와 같이 정의되는 분포를 비중심(noncentral) t 분포라고 하며, δ를 비중심 모수라고 한다.

(a) $\dfrac{\bar{X}-\mu_0}{S/\sqrt{n}} = \dfrac{\bar{X}-\mu+\mu-\mu_0}{\sigma/\sqrt{n}} \Big/ \sqrt{\dfrac{(n-1)S^2}{\sigma^2}/(n-1)}$

$= \left|\dfrac{\bar{X}-\mu}{\sigma/\sqrt{n}} + \dfrac{\mu-\mu_0}{\sigma/\sqrt{n}}\right| \Big/ \sqrt{\dfrac{(n-1)S^2}{\sigma^2}/(n-1)}$

$= |Z+\delta| \Big/ \sqrt{V/(n-1)}$, $Z \sim N(0,1)$, $V \sim \chi^2(n-1)$

(b) $\gamma(|\delta|) = P_{\mu,\sigma^2}\left\{|t| \geq t_{\alpha/2}(n-1)\right\}$

$= P_{\mu,\sigma^2}\left\{\dfrac{\bar{X}-\mu_0}{S/\sqrt{n}} \geq t_{\alpha/2}(n-1)\right\} + P_{\mu,\sigma^2}\left\{\dfrac{\bar{X}-\mu_0}{S/\sqrt{n}} \leq t_{\alpha/2}(n-1)\right\}$ *

suppose that $\dfrac{\bar{X}-\mu_0}{S/\sqrt{n}} > 0$. if $\dfrac{\bar{X}-\mu_0}{S/\sqrt{n}} < 0$, flip the sign + ↔ −

$= P_{\mu,\sigma^2}\left\{\dfrac{Z+\delta}{\sqrt{V/(n-1)}} \geq t_{\alpha/2}(n-1)\right\} + P_{\mu,\sigma^2}\left\{-\dfrac{Z+\delta}{\sqrt{V/(n-1)}} \leq t_{\alpha/2}(n-1)\right\}$

$= P_{\mu,\sigma^2}\left\{\delta \geq t_{\alpha/2}(n-1)\sqrt{V/(n-1)} - Z\right\} + P_{\mu,\sigma^2}\left\{-\delta \leq t_{\alpha/2}(n-1)\sqrt{V/(n-1)} + Z\right\}$

$\gamma(|\delta+\Delta\delta|) = P_{\mu,\sigma^2}\left\{\delta+\Delta\delta \geq t_{\alpha/2}(n-1)\sqrt{V/(n-1)} - Z\right\} + P_{\mu,\sigma^2}\left\{-(\delta+\Delta\delta) \leq t_{\alpha/2}(n-1)\sqrt{V/(n-1)} + Z\right\}$

$\geq P_{\mu,\sigma^2}\left\{\delta \geq t_{\alpha/2}(n-1)\sqrt{V/(n-1)} - Z\right\} + P_{\mu,\sigma^2}\left\{-(\delta+\Delta\delta) \leq t_{\alpha/2}\sqrt{V/(n-1)} + Z\right\}$

∴ $\gamma(|\delta|)$ is an increasing function of $|\delta|$

[Textbook Approach

$P_{\mu,\sigma^2}\left\{|t| \geq t_{\alpha/2}(n-1)\right\} = P\left\{\left|\dfrac{Z+\delta}{\sqrt{V/(n-1)}}\right| \geq t_{\alpha/2}(n-1)\right\}$

$Z \overset{d}{\equiv} -Z \sim N(0,1)$

∴ $\left|\dfrac{Z+\delta}{\sqrt{V/(n-1)}}\right| \overset{d}{\equiv} \left|\dfrac{-Z+\delta}{\sqrt{V/(n-1)}}\right| \overset{d}{\equiv} \left|\dfrac{Z-\delta}{\sqrt{V/(n-1)}}\right|$

∴ $P\left\{\left|\dfrac{Z+\delta}{\sqrt{V/(n-1)}}\right| \geq t_{\alpha/2}(n-1)\right\} = f_{t^{(n)}}(|\delta|) = \gamma(|\delta|)$ 라고 할 수 있다.

$\gamma(|\delta|) = E\left[P\left\{\left|\dfrac{Z+|\delta|}{\sqrt{V/(n-1)}}\right| \geq t_{\alpha/2}(n-1) \,\middle|\, V\right\}\right]$

$P\{|Z+|\delta|| \geq w\} = \bar{\Phi}(w-|\delta|) + \Phi(-w-|\delta|)$,

$w = t_{\alpha/2}(n-1)\sqrt{V/(n-1)}$, $\Phi(z) = \text{Cdf}_{N(0,1)}(z)$, $\bar{\Phi} \equiv 1-\Phi$

$\dfrac{\partial}{\partial\delta} P\{|Z+\delta| \geq w\} = \phi(w+\delta) - \phi(w+\delta) = \phi(w)\, e^{-\frac{1}{2}\delta^2}\left(e^{\delta w} - e^{-\delta w}\right) > 0 \quad \forall \delta > 0, \forall w > 0$

따라서 $\gamma(|\delta|)$는 $|\delta|$의 증가함수이다.

$\equiv \dfrac{\partial}{\partial|\delta|} P\{|Z+|\delta|| \geq w\}$, $|\delta| \to \delta$로 변환

7.7. 지수분포 $\text{Exp}(\theta)$, $0 < \theta < +\infty$ 에서의 랜덤표본 $X_1, \cdots, X_n$을 이용하여

$H_0 : \theta \leq \theta_0$ vs. $H_1 : \theta > \theta_0$

을 검정할 때 유의수준 $\alpha\,(0 < \alpha < 1)$의 최대가능도비 검정을 구하여라.

$f(x;\theta) = \dfrac{1}{\theta} e^{-x/\theta} I_{[0,+\infty)}(x)$

$\log f(x;\theta) = -\log\theta - x/\theta$

$\ell(\theta;x) = \sum_{i=1}^{n} \log f(x_i;\theta) = -n\log\theta - \dfrac{n\bar{x}}{\theta}$

$\hat{\theta}^{\Omega} = \bar{x}$, $\hat{\theta}^{\Omega_0} = \bar{x} \wedge \theta_0$

$2\left\{\ell(\hat{\theta}^{\Omega}) - \ell(\hat{\theta}^{\Omega_0})\right\} = 2\left\{-n\log\left(\dfrac{\hat{\theta}^{\Omega}}{\hat{\theta}^{\Omega_0}}\right) - \dfrac{n\bar{x}}{\hat{\theta}^{\Omega}} + \dfrac{n\bar{x}}{\hat{\theta}^{\Omega_0}}\right\}$

if $\bar{x} \leq \theta_0$: Always accept H_0 because $2\left\{\ell(\hat{\theta}^{\Omega}) - \ell(\hat{\theta}^{\Omega_0})\right\} = 0$

if $\bar{x} > \theta_0$: $2\left\{\ell(\hat{\theta}^{\Omega}) - \ell(\hat{\theta}^{\Omega_0})\right\} = 2\left\{-n\log\left\{\dfrac{\bar{x}}{\theta_0}\right\} - n\cdot\dfrac{\bar{x}}{\bar{x}} + n\dfrac{\bar{x}}{\theta_0}\right\} \geq c$

Rejection region: $\bar{x}/\theta_0 \geq c_1$ or $\bar{x}/\theta_0 \leq c_2$

$X/\theta \sim \text{Exp}(1) \overset{d}{\equiv} \text{Gamma}(1,1)$

$\sum X_i/\theta \sim \text{Gamma}(n,1) \Leftrightarrow 2\sum X_i/\theta \sim \text{Gamma}(n,2) \overset{d}{\equiv} \chi^2(2n)$

∴ Rejection region: $\begin{cases} 2n\bar{x}/\theta_0 \geq c_1 \text{ or } 2n\bar{x}/\theta_0 \leq c_2 \Leftrightarrow \bar{x}/\theta_0 \geq c_1/2n \text{ or } \bar{x}/\theta_0 \leq c_2/2n \\ \int_{c_2}^{c_1} 2n\bar{x}/\theta_0 \cdot \text{pdf}_{\chi^2(2n)}(x)\,dx = 1-\alpha \\ n\log\left\{\dfrac{c_1}{2n}\right\} - \dfrac{c_1}{2} = n\log\left\{\dfrac{c_2}{2n}\right\} - \dfrac{c_2}{2} \end{cases}$

7.8. 확률밀도함수가

$f(x;\theta) = \theta x^{-\theta-1} I_{(1,+\infty)}(x)$

로 주어지는 파레토분포 $\text{Pareto}(1,\theta)$, $\theta > 0$ 에서의 랜덤표본 $X_1, \cdots, X_n$을 이용하여

$H_0 : \theta = 1$ vs $H_1 : \theta \neq 1$

을 검정할 때 유의수준 $\alpha\,(0 < \alpha < 1)$의 최대가능도비 검정을 구하여라.

$\log f(x;\theta) = \log\theta - (\theta+1)\log x$

$\ell(\theta;x) = \sum \log f(x;\theta) = n\log\theta - \theta\, n(\overline{\log x}) - n\overline{\log x}$

$\dot{\ell}(\theta) = \dfrac{n}{\theta} - n(\overline{\log x}) > 0$ ∴ $\hat{\theta}^{MLE} = (\overline{\log x})^{-1}$

$F_X(x) = \int_1^x \theta x^{-\theta-1} dx = \left[-x^{-\theta}\right]_1^x = 1 - x^{-\theta}$

$P(\log x \leq x) = P(x \leq e^x) = 1 - e^{-\theta x}$

∴ $P_Y(y) = \theta e^{-\theta x}$ $Y = \log x$

∴ $\log X \sim \text{Exp}\left\{\theta^{-1}\right\} \overset{d}{\equiv} \text{Gamma}(1,\theta^{-1})$

∴ $2\theta\sum \log X \sim \text{Gamma}(n,2) \sim \chi^2(2n)$

$2\{\ell(\hat{\theta}^{\Omega}) - \ell(\hat{\theta}^{\Omega_0})\} \geq C$

$\Rightarrow 2\{n\log\hat{\theta}^{\Omega} - \hat{\theta}^{\Omega} n(\overline{\log x}) - n\log(1) + n(\overline{\log x}) - n\overline{\log x}\} \geq C$

$= 2\{n\log(\overline{\log x}^{-1}) - (\overline{\log x}^{-1}) n\,\overline{\log x} - n\,\overline{\log x}\} \geq C$

$\therefore$ rejection region : $\overline{\log x} \geq C_1$ or $\overline{\log x} \leq C_2$

$2\theta\Sigma\log x_i \sim \chi^2(2n)$, under H_0, $2n\cdot\overline{\log x} \sim \chi^2(2n)$

$2n\overline{\log x} \geq C_1$ and $2n\overline{\log x} \leq C_2 \Leftrightarrow \overline{\log x} \geq C_1/2n$ or $\overline{\log x} \leq C_2/2n$ ✓✓

$\int_{C_2}^{C_1} 2n\cdot\overline{\log x}\ pdf_{\chi^2(2n)}(x)\,dx = 1-a$ ✓✓

$\log\{C_1/2n\} - C_1/2n = \log\{C_2/2n\} - C_2/2n$ ✓✓

7.9. 확률밀도함수가

$$f(x) = \theta x^{\theta-1} I_{(0,1)}(x)$$

로 주어지는 베타분포 $Beta(\theta,1)$, $\theta>0$ 에서의 랜덤표본 $X_1,\cdots,X_n$을 이용하여

$H_0: \theta \leq 1$ vs $H_1: \theta > 1$

을 검정할 때 유의수준 $a\,(0<a<1)$의 최대가능도비 검정을 구하여라.

$\log f(x_i;\theta) = \log\theta + (\theta-1)\log x$

$\ell(\theta;x) = n\log\theta + (\theta-1)n\,\overline{\log x}$

$\dot{\ell}(\theta;x) = n\theta^{-1} + n\,\overline{\log x} > 0$

$\therefore \hat{\theta}^{MLE} = -\frac{1}{\overline{\log x}} = -(\overline{\log x})^{-1}$

LRT : $2\{\ell(\hat{\theta}^{\Omega}) - \ell(\hat{\theta}^{\Omega_0})\}$

$= 2\{n\log(-\overline{\log x}^{-1}) - (\overline{\log x}^{-1}+1)\cdot n\cdot\overline{\log x} - n\log(\hat{\theta}^{0}) + (\hat{\theta}^{0}-1)\cdot n\,\overline{\log x}\}$

$= 2\{-n\log\{-\overline{\log x}\} - n - n\overline{\log x} - n\log(\hat{\theta}^{0}) + (\hat{\theta}^{0}-1)\cdot n\cdot\overline{\log x}\} \geq C$

$\hat{\theta}^{0}=1$ size of test is largest when $\hat{\theta}^{\Omega_0}=1$ textbook: $\hat{\theta}^{\Omega_0} = \min\{1, -1/\overline{\log x}\}$

LRT : $2\{-n\log\{-\overline{\log x}\} - n - n\,\overline{\log x}\} \geq C$

Rejection region: $-\overline{\log x} \geq C_1$ or $-\overline{\log x} \leq C_2$

$f(x_i;\theta) = \theta x^{\theta-1}$, $F_X(x) = \int_0^x \theta x^{\theta-1}dx = [x^{\theta}]_0^x = x^{\theta}$

$P(-\log X \leq x) = P(X \geq e^{-x}) = 1 - P(X \leq e^{-x}) = 1 - e^{-\theta x}$

$P(-\log X = x) = \theta e^{-\theta x}$

$\therefore -\log X \sim Exp(\theta^{-1}) \overset{d}{\equiv} Gamma(1,\theta^{-1})$

$-2\theta\Sigma\log X_i \sim Gamma(n,2) \overset{d}{\equiv} \chi^2(2n)$

incorrect - nominator & denominator are not indep.

$-2\theta\Sigma\log X_i \sim Gamma(n,2) \overset{d}{\equiv} \chi^2(2n)$

under H_0, $-2\Sigma\log X_i \sim \chi^2(2n)$

$\therefore$ Rejection region = $-2\Sigma\log X_i \geq \chi^2_{1-a}(2n)$ ✓✓

7.10. 두 정규분포 $N(\mu_1,\sigma^2)$, $N(\mu_2,\sigma^2)$, $-\infty<\mu_i<+\infty\ (i=1,2)$, $0<\sigma^2<+\infty$ 에서 서로 독립인 랜덤표본 $X_{11},\cdots,X_{1n_1}$과 $X_{21},\cdots,X_{2n_2}$ $(n_1\geq 2, n_2\geq 2)$를 이용하여

$H_0: \mu_1 \leq \mu_2$ vs. $H_1: \mu_1 > \mu_2$

을 검정할 때 유의수준 $a\,(0<a<1)$의 최대가능도비 검정을 구하여라.

$f(x_i;\theta) = \frac{1}{\sqrt{2\pi}\sigma}\exp\left\{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right\}$

$\log f(x_i;\theta) = -\frac{1}{2}\log 2\pi - \log\sigma - \frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2$

$\ell(\theta;x) = -\frac{n_1+n_2}{2}\log(2\pi) - (n_1+n_2)\log\sigma - \frac{1}{2}\sum_{i=1}^{2}\sum_{j=1}^{n_i}\left(\frac{x_{ij}-\mu_i}{\sigma}\right)^2$

$\theta = (\mu_1,\mu_2,\sigma^2)^t$, $\hat{\theta}^{\Omega} = (\bar{x}_1, \bar{x}_2, \frac{1}{n}\sum_{i=1}^{2}\sum_{j=1}^{n_i}(x_{ij}-\bar{x}_i)^2)$, $\bar{x}_i = \frac{1}{n_i}\sum_{j=1}^{n_i} x_{ij}$

if $\bar{x}_1 \leq \bar{x}_2$: $\hat{\theta}^{\Omega_0} = (\bar{x}_1, \bar{x}_2, \frac{1}{n}\sum_{i=1}^{2}\sum_{j=1}^{n_i}(x_{ij}-\bar{x}_i)^2)$

otherwise, $\hat{\theta}^{\Omega_0} = \left(\frac{n_1\bar{x}_1+n_2\bar{x}_2}{n_1+n_2}, \frac{n_1\bar{x}_1+n_2\bar{x}_2}{n_1+n_2}, \frac{1}{n}\sum_{i=1}^{2}\sum_{j=1}^{n_i}(x_{ij}-\hat{\mu}_i)^2\right)$

(derivation of $\hat{\theta}^{\Omega_0}$)

$\underset{\mu_1,\mu_2\in\Omega_0}{\operatorname{argmax}}\ \ell(\theta;x) = \underset{\mu_1,\mu_2\in\Omega_0}{\operatorname{argmin}} \Sigma(x_{1j}-\mu_1)^2 + \Sigma(x_{2j}-\mu_2)^2$

$= \underset{}{\operatorname{argmin}} \Sigma(x_{1j}-\bar{x}_1) + n_1(\bar{x}_1-\mu_1)^2 + \Sigma(x_{2j}-\bar{x}_2)^2 + n_2(\bar{x}_2-\mu_2)^2$

$= \underset{}{\operatorname{argmin}}\ n_1(\bar{x}_1-\mu_1)^2 + n_2(\bar{x}_2-\mu_2)^2$

$= \underset{\mu_i\in\Omega}{\operatorname{argmin}}\ n_1(\bar{x}_1-\mu_1)^2 + n_2(\bar{x}_2-\mu_1)^2 = \underset{\mu_2\in\Omega}{\operatorname{argmax}}\ n_1(\bar{x}_1-\mu_2)^2 + n_2(\bar{x}_2-\mu_2)^2$

$= \underset{\mu_i\in\Omega}{\operatorname{argmin}}\ g(\mu_1)$, $\frac{\partial g}{\partial\mu_1} = -2n_1(\bar{x}_1-\mu_1) - 2n_2(\bar{x}_2-\mu_1) = 0$

$(n_1+n_2)\mu_1 = n_1\bar{x}_1 + n_2\bar{x}_2 \ \therefore \hat{\mu}_1 = \frac{n_1\bar{x}_1+n_2\bar{x}_2}{n_1+n_2}$

$\frac{\partial^2 g}{\partial\mu_1^2} = 2n_1+2n_2 > 0 \ \therefore$ function $g(\mu_1)$ is strictly convex

$2\{\ell(\hat{\theta}^{\Omega};x) - \ell(\hat{\theta}^{\Omega_0};x)\} \geq C \Leftrightarrow \log(\hat{\sigma}^{\Omega_0}/\hat{\sigma}^{\Omega}) \geq C \Leftrightarrow \hat{\sigma}^{\Omega_0}/\hat{\sigma}^{\Omega} \geq C$

$\hat{\sigma}^{\Omega_0} = \frac{1}{n}\sum_{i=1}^{2}\sum_{j=1}^{n_i}(x_{ij}-\bar{x})^2$, $\hat{\sigma}^{\Omega} = \frac{1}{n}\sum_{i=1}^{2}\sum_{j=1}^{n_i}(x_{ij}-\bar{x}_i)^2$ $\bar{x} = \frac{1}{n_1+n_2}\sum_{i=1}^{2}\sum_{j=1}^{n_i} x_{ij}$

$n\hat{\sigma}^{\Omega_0}/\sigma \sim \chi^2(n-1)$, $n\hat{\sigma}^{\Omega}/\sigma \sim \chi^2(n_1+n_2-2) \overset{d}{\equiv} \chi^2(n-2)$

$\left\{\hat{\sigma}^{\Omega_0}/(n-1)\right\} / \left\{\hat{\sigma}^{\Omega}/(n-a)\right\} \sim F(n-1, n-a)$

test: reject H_0 if $\hat{\sigma}^{\Omega_0}/\hat{\sigma}^{\Omega} \geq F_{1-a}(n-1,n-a)$ or equivalently,

$\hat{\sigma}^{\Omega}/\hat{\sigma}^{\Omega_0} \leq F_a(n-a,n-1)$

$$\log\left(\hat{\sigma}^{2(0)}/\hat{\sigma}^{2}\right) = \log\left(\frac{\frac{1}{n}\sum_{i=1}^{2}\sum_{j=1}^{n_i}(x_{ij}-\bar{x})^2}{\frac{1}{n}\sum_{i=1}^{2}\sum_{j=1}^{n_i}(x_{ij}-\bar{x}_i)^2}\right) =$$

$$\star\ \frac{\sum\sum(x_{ij}-\bar{x}_i+\bar{x}_i-\bar{x})^2}{\sum\sum(x_{ij}-\bar{x}_i)^2} = \frac{\sum\sum(x_{ij}-\bar{x}_i)+\sum\sum(\bar{x}_i-\bar{x})^2}{\sum\sum(x_{ij}-\bar{x}_i)^2} = 1+\frac{\sum\sum\left(\bar{x}_i-\frac{n_1\bar{x}_1+n_2\bar{x}_2}{n_1+n_2}\right)^2}{\sum\sum(x_{ij}-\bar{x}_i)^2}$$

$$= 1+\frac{n_1\left(\bar{x}_1-\frac{n_1\bar{x}_1+n_2\bar{x}_2}{n_1+n_2}\right)^2+n_2\left(\bar{x}_2-\frac{n_1\bar{x}_1+n_2\bar{x}_2}{n_1+n_2}\right)^2}{\sum\sum(x_{ij}-\bar{x}_i)^2}$$

$$= 1+\frac{n_1\left(\frac{n_2\bar{x}_1-n_2\bar{x}_1}{n_1+n_2}\right)^2+n_2\left(\frac{n_1\bar{x}_1-n_1\bar{x}_2}{n_1+n_2}\right)^2}{\sum\sum(x_{ij}-\bar{x}_i)^2} = 1+\frac{\frac{n_1n_2}{n_1+n_2}(\bar{x}_1-\bar{x}_2)^2}{\sum\sum(x_{ij}-\bar{x}_i)^2}$$

test shape

$$\Rightarrow \frac{\bar{x}_1-\bar{x}_2}{\sqrt{n_1^{-1}+n_2^{-1}}\,S_p} \geq C \qquad S_p = \sum\sum(x_{ij}-\bar{x}_i)^2/(n_1+n_2-2)$$

기각역의 확률은 $\gamma(\mu_1,\mu_2,\sigma^2) = P_{\mu_1,\mu_2,\sigma^2}\left(\frac{\bar{x}_1-\bar{x}_2}{S_p\sqrt{n_1^{-1}+n_2^{-1}}} \geq C\right)$

로 주어진다. 한편 표본분포 이론으로부터

$$\frac{\bar{x}_1-\bar{x}_2-(\mu_1-\mu_2)}{\sqrt{n_1^{-1}+n_2^{-1}}\,S_p} \sim t(n_1+n_2-2)$$ 이므로, <예 7.2.2>에서와 같은 방법으로

이 기각역의 확률이 $\frac{\mu_1-\mu_2}{\sqrt{n_1^{-1}+n_2^{-1}}\cdot S_p}$ 의 증가함수임을 알수 있다. 따라서 귀무가설

$H_0: \mu_1 \leq \mu_2$ 하에서 이 기각역의 확률이 최대가 되는 것은 $\mu_1=\mu_2$일 때이다.

또한 $\mu_1=\mu_2$일 때

$$\frac{\bar{x}_1-\bar{x}_2}{\sqrt{n_1^{-1}+n_2^{-1}}\cdot S_p} \sim t(n_1+n_2-2)$$

이므로, 위의 기각역에서 $C=t_\alpha(n_1+n_2-2)$로 하면 이러한 검정은 크기가 α인 유의수준 α의 검정이다. 즉 유의수준 $\alpha\ (0<\alpha<1)$의 최대가능도비 검정의 기각역은 다음과 같다:

$$\frac{\bar{x}_1-\bar{x}_2}{\sqrt{n_1^{-1}+n_2^{-1}}\,S_p} \geq t_\alpha(n_1+n_2-2)$$

7.11. 두 정규분포 $N(\mu_1,\sigma^2)$, $N(\mu_2,\sigma^2)$, $-\infty<\mu_i<+\infty\ (i=1,2)$ $0<\sigma^2<+\infty$ 에서 서로 독립인 랜덤표본 $X_{11},\cdots,X_{1n_1}$과 $X_{21},\cdots,X_{2n_2}\ (n_1\geq 2, n_2\geq 2)$를 이용하여

$$H_0: \mu_1=\mu_2 \quad vs \quad H_1: \mu_1\neq\mu_2$$

을 검정할 때 사용하는 <예 7.2.4>에서의 양쪽검정, 즉 기각역이

$$\left|\frac{\bar{x}_1-\bar{x}_2}{\sqrt{n_1^{-1}+n_2^{-1}}\,S_p}\right| \geq t_{\alpha/2}(n_1+n_2-2)$$

로 주어진 검정의 검정력함수는 $|\delta| = \left|\frac{\mu_1-\mu_2}{\sigma\sqrt{n_1^{-1}+n_2^{-1}}}\right|$ 의 증가함수임을 밝혀라.

$$\gamma(\theta) = P_{\theta_1}\left\{\left|\frac{\bar{x}_1-\bar{x}_2}{\sqrt{n_1^{-1}+n_2^{-1}}\,\sigma}\right| \Big/ \sqrt{\frac{(n_1+n_2-2)S_p^2}{\sigma^2}/(n_1+n_2-2)} \geq t_{\alpha/2}(n_1+n_2-1)\right\}$$

let $V=\frac{(n_1+n_2-2)S_p^2}{\sigma^2}$, $V\sim\chi^2(n_1+n_2-2)$, $\frac{\bar{x}_1-\bar{x}_2}{\sqrt{n_1^{-1}+n_2^{-1}}\,\sigma} = Z+\delta$, $Z\sim N(0,1)$

$$\Rightarrow P_{\theta_1}\left\{\left|\frac{(\bar{x}_1-\bar{x}_2)}{\sqrt{n_1^{-1}+n_2^{-1}}\cdot\sigma}\right| \Big/ \sqrt{V/(n_1+n_2-2)} \geq t_{\alpha/2}(n_1+n_2-1)\right\}$$

$$\therefore \gamma(\delta) = P_\theta\left\{\left|Z+\delta\right| \geq t_{\alpha/2}(n_1+n_2-1)\cdot\sqrt{V/(n_1+n_2-2)}\right\}$$

Z is symmetrical. $\therefore |Z+\delta| = |-Z+\delta| = |Z-\delta| = |Z+|\delta||$

Let $w = t_{\alpha/2}(n_1+n_2-2)\sqrt{V/(n_1+n_2-2)}$.

$$\gamma(\delta) = P_\theta\{Z+|\delta| \geq w\} + P_\theta\{Z+|\delta| \leq -w\}$$
$$= 1-\Phi(w-|\delta|)+\Phi(-w-|\delta|)$$

$$\frac{\partial\gamma(\delta)}{\partial|\delta|} = \phi(w-|\delta|)-\phi(-w-|\delta|) = \phi(w-|\delta|)-\phi(w+|\delta|)$$
$$= \frac{1}{\sqrt{2\pi}}\exp\left\{-\tfrac{1}{2}(w-|\delta|)^2\right\} - \frac{1}{\sqrt{2\pi}}\exp\left\{-\tfrac{1}{2}(w+|\delta|)^2\right\}$$
$$= \frac{1}{\sqrt{2\pi}}\exp\left\{-\tfrac{1}{2}w^2-\tfrac{1}{2}\delta^2\right\}\left[e^{w|\delta|}-e^{-w|\delta|}\right]$$
$$= \phi(w)\exp\left\{-\tfrac{1}{2}\delta^2\right\}\left(e^{w|\delta|}-e^{-w|\delta|}\right) > 0 \quad \forall\delta>0,\ \forall w>0$$

$\forall|\delta|>0$?

$\therefore \gamma(|\delta|)$ is an increasing function of $|\delta|$

☆ 7.12. 두 정규분포 $N(\mu_1,\sigma_1^2)$, $N(\mu_2,\sigma_2^2)$, $-\infty<\mu_i<+\infty$, $0<\sigma_i^2<+\infty$ $(i=1,2)$ 에서 서로 독립인 랜덤표본 $X_{11},\cdots,X_{1n_1}$ 과 $X_{21},\cdots,X_{2n_2}$ $(n_1\geq 2, n_2\geq 2)$를 이용하여

$$H_0:\sigma_1^2\leq\sigma_2^2 \quad vs \quad H_1:\sigma_1^2>\sigma_2^2$$

을 검정할 때 유의수준 α $(0<\alpha<1)$ 의 최대가능도비 검정을 구하여라.

$$\log f(x;\theta)=-\frac{1}{2}\log(2\pi)-\log(\sigma_i)-\frac{1}{2}\left(\frac{x-\mu_i}{\sigma_i}\right)^2$$

$$\ell(\theta;x)=\sum_{i=1}^{2}\sum_{j=1}^{n_i}\log f(x_{ij};\theta)$$

$$=-\frac{n}{2}\log(2\pi)-n_1\log(\sigma_1)-n_2\log(\sigma_2)-\frac{1}{2}\sum_{j=1}^{n_1}\left(\frac{x_{1j}-\mu_1}{\sigma_1}\right)^2-\frac{1}{2}\sum_{j=1}^{n_2}\left(\frac{x_{2j}-\mu_2}{\sigma_2}\right)^2$$

$$\hat{\mu}_1^{\Omega}=\frac{1}{n_1}\sum_{j=1}^{n_1}x_{1j},\quad \hat{\mu}_2^{\Omega}=\frac{1}{n_2}\sum_{j=1}^{n_2}x_{2j},\quad \hat{\sigma}_1^{2\,\Omega}=\frac{1}{n_1}\sum_{j=1}^{n_1}(x_{1j}-\bar{x}_1)^2,\quad \hat{\sigma}_2^{2\,\Omega}=\frac{1}{n_2}\sum_{j=1}^{n_2}(x_{2j}-\bar{x}_2)^2$$

$$\hat{\mu}_1^{\Omega_0}=\bar{x}_1,\quad \hat{\mu}_2^{\Omega_0}=\bar{x}_2,\quad \hat{\sigma}_1^{2\,\Omega_0},\quad \hat{\sigma}_2^{2\,\Omega_0}$$

$$2\left\{\ell(\hat{\theta}^{\Omega};x)-\ell(\hat{\theta}^{\Omega_0};x)\right\}\geq c \Leftrightarrow -n_1\log\left(\hat{\sigma}_1^2/\hat{\sigma}_1^{2\,\Omega_0}\right)-n_2\log\left(\hat{\sigma}_2^2/\hat{\sigma}_2^{2\,\Omega_0}\right)\geq c$$

$$\Leftrightarrow n_1\log\left(\hat{\sigma}_1^{2\,0}/\hat{\sigma}_1^2\right)+n_2\log\left(\hat{\sigma}_2^{2\,0}/\hat{\sigma}_2^2\right)\geq c$$

$$\hat{\sigma}_1^{2\,\Omega_0}=\hat{\sigma}_2^{2\,\Omega_0}=\frac{n_1\hat{\sigma}_1^2+n_2\hat{\sigma}_2^2}{n_1+n_2},$$

$$\hat{\sigma}_1^{2\,0}/\hat{\sigma}_1^2=\frac{n_1}{n_1+n_2}+\frac{n_2}{n_1+n_2}\frac{\hat{\sigma}_2^2}{\hat{\sigma}_1^2},\quad \hat{\sigma}_2^{2\,0}/\hat{\sigma}_2^2=\frac{n_2}{n_1+n_2}+\frac{n_1}{n_1+n_2}\frac{\hat{\sigma}_1^2}{\hat{\sigma}_2^2}$$

rejection region: $n_1\log\left(\frac{n_1}{n_1+n_2}+\frac{n_2}{n_1+n_2}t\right)+n_2\log\left(\frac{n_2}{n_1+n_2}+\frac{n_1}{n_1+n_2}t^{-1}\right)\geq c$

$$,\ t=\hat{\sigma}_2^2/\hat{\sigma}_1^2$$

$$\Rightarrow \frac{\hat{\sigma}_1^2}{\hat{\sigma}_2^2}\geq c$$

under H_0, the size of test is largest when $\sigma_1=\sigma_2=\sigma$

$$\therefore \frac{\hat{\sigma}_1^2}{\hat{\sigma}_2^2}\geq c \Leftrightarrow \frac{\hat{S}_1^2}{\hat{S}_2^2}\geq c \Rightarrow \frac{\sqrt{\frac{(n_1-1)S_1^2}{\sigma^2}/(n_1-1)}}{\sqrt{\frac{(n_2-1)S_2^2}{\sigma^2}/(n_2-1)}}\overset{d}{\equiv}\frac{V_1/(n_1-1)}{V_2/(n_2-1)}\sim F(n_1-1,n_2-1)$$

Here, $V_1\sim\chi^2(n_1-1)$, $V_2\sim\chi^2(n_2-1)$, $S_1^2=\frac{1}{n_1-1}\sum_{j=1}^{n_1}(x_{1j}-\bar{x}_1)^2$, $S_2^2=\frac{1}{n_2-1}\sum_{j=1}^{n_1}(x_{2j}-\bar{x}_2)^2$

$\therefore$ Reject H_0 if $\hat{S}_1^2/\hat{S}_2^2\geq F_\alpha(n_1-1,n_2-1)$ ✓✓

* Textbook Approach review
* if $\sigma_1^2=\sigma_2^2\Rightarrow S_1^2/S_2^2\sim F(n_1-1,n_2-1)$: memorize!

☆ 7.13. 확률밀도함수

$$f(x;\mu,\sigma)=\frac{1}{\sigma}\exp\left\{-\frac{x-\mu}{\sigma}\right\}I_{[\mu,+\infty)}(x)$$

로 주어지는 두 개의 모수를 갖는 지수분포 $Exp(\mu,\sigma)$, $-\infty<\mu<+\infty$, $\sigma>0$ 에서의 랜덤표본 $X_1,\cdots,X_n$ 을 이용하여

$$H_0:\mu=\mu_0 \quad vs \quad H_1:\mu\neq\mu_0 \ (\mu_0 \text{ is given})$$

을 검정할 때 유의수준 α $(0<\alpha<1)$ 의 최대가능도비 검정을 구하여라.

$$\log f(x;\theta)=-\log\sigma-\frac{1}{\sigma}(x-\mu)$$

$$\ell(\theta;x)=-n\log\sigma-\frac{1}{\sigma}n(\bar{x}-\mu)$$

$$\hat{\mu}^{\Omega}=x_{(1)}=\min_{1\leq i\leq n}x_i,\quad \frac{\partial}{\partial\sigma}\ell(\theta;x)=-\frac{n}{\sigma}+\frac{1}{\sigma^2}n(\bar{x}-\mu)>0$$

$$\Rightarrow(\bar{x}-\hat{\mu}^{\Omega})>\hat{\sigma}^{\Omega}$$

$$\therefore \hat{\sigma}^{\Omega}=\bar{x}-x_{(1)}$$

$$\therefore \hat{\theta}^{\Omega}=(\hat{\mu}^{\Omega},\hat{\sigma}^{\Omega})^t=(x_{(1)},\bar{x}-x_{(1)})^t$$
$$\hat{\theta}^{\Omega_0}=(\hat{\mu}^{\Omega_0},\hat{\sigma}^{\Omega_0})^t=(\mu_0,\bar{x}-\mu_0)^t$$

$$2\left\{\ell(\hat{\theta}^{\Omega})-\ell(\hat{\theta}^{\Omega_0})\right\}\geq c \Rightarrow\left[-n\log\hat{\sigma}-\frac{1}{\hat{\sigma}}n(\bar{x}-\hat{\mu})\right]-\left[-n\log\hat{\sigma}^0-\frac{1}{\hat{\sigma}^0}n(\bar{x}-\hat{\mu}^0)\right]\geq c$$

$$\Rightarrow\left[n\log(\hat{\sigma}^0/\hat{\sigma})\right]\geq c$$

$$\Rightarrow n\log\left(\bar{x}-\mu_0/\bar{x}-x_{(1)}\right)\geq c$$

$$(X-\mu)/\sigma\sim Exp(1)\equiv Gamma(1,1)$$
$$\sum(X_i-\mu)/\sigma\sim Gamma(n,1)$$

$$\sum X_i=X_{(1)}+X_{(2)}+\cdots+X_{(n)}=nX_{(1)}+(n-1)(X_{(2)}-X_{(1)})+\cdots+X_{(n)}$$

$$(n-r+1)X_{(r)}\overset{iid}{\sim}Exp(1),\ r=1,2,\cdots,n$$

$$n\log\left(\frac{\bar{x}-\mu_0}{\bar{x}-x_{(1)}}\right)=n\log\left(\frac{\bar{x}-x_{(1)}+x_{(1)}-\mu_0}{\bar{x}-x_{(1)}}\right)$$

$$=n\log\left(1+\frac{(x_{(1)}-\mu_0)/\sigma}{(\bar{x}-x_{(1)})/\sigma}\right)$$

$$=n\log\left\{1+\frac{n(x_{(1)}-\mu_0)/\sigma}{\frac{\sum x_i-n\mu_0}{\sigma}-\frac{n(x_{(1)}-\mu_0)}{\sigma}}\right\}\geq c$$

$$n(x_{(1)}-\mu_0)/\sigma\sim Exp(1)$$
$$(\sum x_i-n\mu_0)/\sigma\sim Gamma(n,1)$$

$$\therefore 2n(x_{(1)}-\mu_0)/\sigma\sim Gamma(1,2)\overset{d}{\equiv}\chi^2(2)$$
$$2\left\{(\sum x_i-n\mu_0)/\sigma-n(x_{(1)}-\mu_0)/\sigma\right\}\sim Gamma(n-1,2)\overset{d}{\equiv}\chi^2(2n-2)$$

$$\therefore \frac{\left\{(x_{(1)}-\mu_0)/\sigma\right\}/2}{\left\{(\bar{x}-x_{(1)})/\sigma\right\}/(2n-2)}=\frac{V_1/n_1}{V_2/n_2}\sim F(n_1,n_2)\equiv F(2,2n-2)$$

$$\therefore \text{Critical region}:\ \frac{(x_{(1)}-\mu_0)/2\sigma}{(\bar{x}-x_{(1)})/(2n-2)\sigma}\geq F_\alpha(2,2n-2)$$ ✓✓

or $x_{(1)}<\mu_0$

$+\ (x_{(1)}-\mu_0)/2\sigma \perp\!\!\!\perp (\bar{x}-x_{(1)})/(2n-2)\sigma$

7.14. 연습문제 (7.13)에서 가설

$H_0: \sigma = \sigma_0$ vs $H_1: \sigma \neq \sigma_0$ (σ_0는 주어진 값)

을 검정할 때 유의수준 $\alpha\ (0<\alpha<1)$의 최대가능도비 검정을 구하여라.

$$f(x;\theta) = \frac{1}{\sigma} e^{-\frac{x-\mu}{\sigma}} I_{[\mu,+\infty)}(x)$$

$$\log f(x;\theta) = -\log\sigma - \frac{x-\mu}{\sigma} + \log\left(I_{[\mu,+\infty)}(x)\right)$$

$$\ell(\theta;x) = -n\log\sigma - \frac{n(\bar{x}-\mu)}{\sigma} + n\log\left(I_{[\mu,+\infty)}(x)\right)$$

$$\ell(\hat{\theta}^{\Omega};x) - \ell(\hat{\theta}^{\Omega_0};x) = -n\log\left(\hat{\sigma}^{\Omega}/\hat{\sigma}^{\Omega_0}\right) - \frac{n(\bar{x}-x_{(1)})}{\hat{\sigma}^{\Omega}} + \frac{n(\bar{x}-x_{(1)})}{\hat{\sigma}^{\Omega_0}}$$

$$= -n\log\left((\bar{x}-x_{(1)})/\sigma^{\Omega_0}\right) - n + \frac{n(\bar{x}-x_{(1)})}{\hat{\sigma}^{\Omega_0}} \geq c$$

$$\therefore \text{ critical region: } \frac{\bar{x}-x_{(1)}}{\sigma^{\Omega_0}} \geq c_1 \quad \frac{\bar{x}-x_{(1)}}{\sigma^{\Omega_0}} \leq c_2$$

under $H_0: \sigma^{\Omega_0} = \sigma$, critical region:

$$\therefore \frac{\bar{x}-x_{(1)}}{\sigma} \geq c_1 \quad \frac{\bar{x}-x_{(1)}}{\sigma} \leq c_2$$

$$\frac{n(\bar{x}-x_{(1)})}{\sigma} = \frac{\sum x_{(i)} - nx_{(1)}}{\sigma} \sim \text{Gamma}(n-1,1)$$

$$\because (n-r+1)x_{(1)} \overset{iid}{\sim} \text{Exp}(1) \quad (i=1,2,\cdots,r)$$

$$\therefore 2n(\bar{x}-x_{(1)})/\sigma \sim \chi^2(2n-2)$$

critical region:

① $\int_{2nc_1}^{2nc_2} pdf_{\chi^2(2n-2)}(x)\,dx = 1-\alpha$

② $-n\log c_1 - n + nc_1 = -n\log c_2 - n + nc_2$ ✓✓

$\Updownarrow$

$\log c_1 - c_1 = \log c_2 - c_2$

③ $\frac{\bar{x}-x_{(1)}}{\sigma^{\Omega_0}} \geq c_1 \quad \frac{\bar{x}-x_{(1)}}{\sigma^{\Omega_0}} \leq c_2$

Not too hard. Good practice

☆ 7.15 균등분포 $U[0,\theta]$, $\theta>0$ 에서의 랜덤표본 $X_1,\cdots,X_n$을 이용하여 다음 가설을 검정할 때 유의수준 $\alpha\ (0<\alpha<1)$의 최대가능도비 검정을 구하여라

(a) $H_0: \theta = \theta_0$ vs $H_1: \theta \neq \theta_0$ (θ_0는 주어진 값)

(b) $H_0: \theta \geq \theta_0$ vs $H_1: \theta < \theta_0$ (θ_0는 주어진 값)

$$f_X(x;\theta) = \frac{1}{\theta} I_{[0,\theta]}(x) \qquad \log f_X(x) = -\log\theta + \log I_{[0,\theta]}(x)$$

$$\ell(\theta;X) = \sum_{i=1}^{n} \log f_X(x_i;\theta) = -n\log\theta + \log I_{(x_{(n)},\infty)}(\theta)$$

$$\ell(\hat{\theta}^{\Omega};x) - \ell(\hat{\theta}^{\Omega_0};x)$$

$$= -n\log\left(\hat{\theta}^{\Omega}/\hat{\theta}^{\Omega_0}\right) + \log I_{(x_{(n)},\infty)}(\theta) - \log I_{(x_{(n)},\infty)}(\theta)$$

$$= -n\log\left(x_{(n)}/\hat{\theta}^{\Omega_0}\right) + \cdots \geq c$$

$$\therefore \text{ critical region: } \frac{x_{(n)}}{\hat{\theta}^{\Omega}} \leq c$$

under H_0, $X/\theta_0 \sim U[0,1]$ $\therefore (X/\theta_0)_{(n)} = X_{(n)}/\theta_0 \sim \text{Beta}(n,1)$

critical region:
$$\begin{cases} 1 & \text{if } x_{(n)} > \theta_0 \text{ or } x_{(n)}/\theta_0 \leq c \\ \int_c^1 pdf_{\text{Beta}(n,1)}(x)\,dx = 1-\alpha & \end{cases}$$

$c = \text{Beta}_{\alpha}(n,1)$

✓✓ equivalent answer

$P_{\theta_0}(X_n/\theta_0 \leq c) + P_{\theta_0}(X_{(n)} > \theta_0) = \alpha$ *

$\Updownarrow$

$\therefore c^n = \alpha$

(b) $H_0: \theta \geq \theta_0$ vs $H_1: \theta < \theta_0$ (θ_0 is given)

Critical region shape: $g(x) < c$

$X_{(n)}/\theta$ is a decreasing function of θ

$$\therefore \max_{\theta \in \Omega_0} P(X_{(n)}/\theta \leq c) = P(X_{(n)}/\theta_0 \leq c) = \alpha$$

$\therefore$ answer is the same as (b)

* under $H_0: \theta = \theta_0$

$X_{(n)}/\theta_0 \sim \text{Beta}(n,1)$

$\therefore P(X_n > \theta_0)$

$\Updownarrow$

$P(X_n/\theta > 1) = 0$

$\because$ range of Beta(·,·) is $(0,1)$

7.16. $K\,(K\geq 3)$개의 베르누이 분포 $\text{Bernoulli}(p_i)$ $0\leq p_i\leq 1$ $(i=1,2,\cdots,K)$에서 서로 독립인 랜덤표본 $X_{i1},\cdots,X_{in_i}$를 이용하여 다음 가설을 검정하려고 할 때 다음에 답하여라.

$$H_0:\ p_1=\cdots=p_K \quad \text{vs} \quad H_1:\ p_1,\cdots,p_K \text{ 같지 않다 (i.e., not } H_0)$$

(a) 최대가능도비 검정통계량이 다음과 같이 주어진 것을 밝혀라.

$$2\left\{\ell(\hat{p})-\ell(\hat{p}^0)\right\}=2\sum_{i=1}^{K}\left\{n_i\hat{p}_i\log(\hat{p}_i/\hat{p}_i^0)+n_i\hat{q}_i\log(\hat{q}_i/\hat{q}_i^0)\right\}$$

$$\hat{p}_i=\sum_{j=1}^{n_i}x_{ij}/n_i=1-\hat{q}_i,\quad \hat{p}_i^0=\sum_{i=1}^{K}\sum_{j=1}^{n_i}x_{ij}/n=1-q^0,\quad n=n_1+\cdots+n_K$$

(b) 표본크기에 대하여

$$n_i\longrightarrow\infty,\quad n_i/(n_1+\cdots+n_K)\longrightarrow\gamma_i\ \ (0<\gamma_i<1)\quad (i=1,\cdots,K)$$

이 만족된다고 할 때, 귀무가설 H_0하에서 최대가능도 검정통계량의 다음과 같은 근사에 대하여 설명하려라.

$$2\left\{\ell(\hat{p})-\ell(\hat{p}^0)\right\}\simeq\sum_{i=1}^{K}\left(\frac{X_{i\cdot}-n_i\hat{p}_i^0}{\sqrt{n_i\hat{p}_i^0\hat{q}_i^0}}\right)^2,\quad X_{i\cdot}=\sum_{j=1}^{n_i}X_{ij}$$

(c) (b)로부터 최대가능도비 검정통계량이 귀무가설 H_0 하에서 근사적으로 $\chi^2(K-1)$ 분포를 따르는 것을 설명하여라.

(a) $X_{ij}\sim\text{Bernoulli}(p_i)$, $1\leq i\leq r$, $1\leq j\leq c$

$f_X(x_{ij})=p_i^{x_{ij}}(1-p_i)^{1-x_{ij}}$

$\log f(x_{ij})=x_{ij}\log(p_i)+(1-x_{ij})\log(1-p_i)$

$\ell(\theta)=\sum_{i=1}^{r}\sum_{j=1}^{c}\log f(x_{ij})=\sum_{i=1}^{r}\sum_{j=1}^{c}x_{ij}\log(p_i)+(1-x_{ij})\log(1-p_i)$

$$\frac{\partial\ell}{\partial p_i}=\sum_{j=1}^{c}\frac{x_{ij}}{p_i}-\frac{1-x_{ij}}{1-p_i}>0$$

$$=\frac{x_{i\cdot}}{p_i}-\frac{c-x_{i\cdot}}{1-p_i}>0,\quad x_{i\cdot}=\sum_{j=1}^{c}x_{i\cdot}$$

$$\Leftrightarrow (1-p_i)x_{i\cdot}-p_i(c-x_{i\cdot})>0$$

$$\Rightarrow x_{i\cdot}-x_{i\cdot}p_i-p_ic+p_ix_{i\cdot}>0$$

$$\Rightarrow p_i<\frac{x_{i\cdot}}{c}=\frac{1}{n_i}\sum_{j=1}^{n_i}x_{ij}=\overline{x_i}\ \ (=\frac{x_{i\cdot}}{n_i})$$

$$\therefore\ \hat{p}_i=\frac{x_{i\cdot}}{n_i}$$

under H_0, $\ell(\theta)=\sum_{i=1}^{r}\sum_{j=1}^{c}x_{ij}\log(p)+(1-x_{ij})\log(1-p)$

$$\frac{\partial\ell}{\partial p}=\sum_{i=1}^{r}\sum_{j=1}^{c}\frac{x_{ij}}{p}-\frac{1-x_{ij}}{1-p}=\sum_{i=1}^{r}\sum_{j=1}^{c}(1-p)x_{ij}-p(1-x_{ij})>0$$

$$\Leftrightarrow\sum_{i=1}^{r}\sum_{j=1}^{c}x_{ij}-p>0$$

$$\therefore\ \hat{p}^0=\frac{1}{n}\sum_{i=1}^{K}\sum_{j=1}^{n_i}x_{ij}\quad (r=K,\ c=n_i)$$

$$\ell(\hat{\theta})-\ell(\hat{\theta}^0)=\sum_{i=1}^{r}\sum_{j=1}^{c}x_{ij}\log\left(\frac{\hat{p}_i}{\hat{p}_i^0}\right)+(1-x_{ij})\log\left(\frac{1-\hat{p}_i}{1-\hat{p}_i^0}\right)$$

$$=\sum_{i=1}^{r}x_{i\cdot}\log\left(\frac{\hat{p}_i}{\hat{p}_i^0}\right)+(n_i-x_{i\cdot})\log\left(\frac{1-\hat{p}_i}{1-\hat{p}_i^0}\right)$$

$$=\sum_{i=1}^{r}n_i\hat{p}_i\log\left(\frac{\hat{p}_i}{\hat{p}_i^0}\right)+n_i(1-\hat{p}_i)\log\left(\frac{1-\hat{p}_i}{1-\hat{p}_i^0}\right)$$

$$=\sum_{i=1}^{r}n_i\hat{p}_i^0\left(\frac{\hat{p}_i}{\hat{p}_i^0}\right)\log\left(\frac{\hat{p}_i}{\hat{p}_i^0}\right)+n_i(1-\hat{p}_i^0)\left(\frac{1-\hat{p}_i}{1-\hat{p}_i^0}\right)\log\left(\frac{1-\hat{p}_i}{1-\hat{p}_i^0}\right)$$

$t\log t\simeq(t-1)+\frac{1}{2}(t-1)^2$

$$\simeq\sum_{i=1}^{r}n_i\hat{p}_i^0\left\{\left(\frac{\hat{p}_i}{\hat{p}_i^0}-1\right)+\frac{1}{2}\left(\frac{\hat{p}_i}{\hat{p}_i^0}-1\right)^2\right\}+n_i(1-\hat{p}_i^0)\left\{\left(\frac{1-\hat{p}_i}{1-\hat{p}_i^0}-1\right)+\frac{1}{2}\left(\frac{1-\hat{p}_i}{1-\hat{p}_i^0}-1\right)^2\right\}$$

$$=\sum_{i=1}^{r}\underbrace{n_i\hat{p}_i-n_i\hat{p}_i^0}_{=0}+\frac{n_i\hat{p}_i^0}{2}\left(\frac{\hat{p}_i-\hat{p}_i^0}{\hat{p}_i^0}\right)^2+\underbrace{n_i(1-\hat{p}_i)-n_i(1-\hat{p}_i^0)}_{=0}+\frac{n_i(1-\hat{p}_i^0)}{2}\left(\frac{\hat{p}_i-\hat{p}_i^0}{1-\hat{p}_i^0}\right)^2$$

$$=\frac{1}{2}\sum_{i=1}^{r}\frac{(n_i\hat{p}_i-n_i\hat{p}_i^0)^2}{np_i^0}+\frac{(n_i\hat{p}_i-n_i\hat{p}_i^0)^2}{n_i(1-\hat{p}_i^0)}$$

$$=\frac{1}{2}\sum_{i=1}^{r}\frac{(n_i\hat{p}_i-n_i\hat{p}_i^0)^2(1-\hat{p}_i^0)+(n_i\hat{p}_i-n_i\hat{p}_i^0)^2p_i^0}{n\hat{p}_i^0(1-\hat{p}_i^0)}$$

$$=\frac{1}{2}\sum_{i=1}^{r}\frac{(n_i\hat{p}_i-n_i\hat{p}_i^0)^2}{np_i^0(1-\hat{p}_i^0)}=\frac{1}{2}\sum_{i=1}^{K}\left(\frac{n_i\hat{p}_i-n_i\hat{p}_i^0}{\sqrt{n_i\hat{p}_i^0\hat{q}_i^0}}\right)^2$$

$$\therefore\ 2\left(\ell(\hat{\theta})-\ell(\hat{\theta}^0)\right)=\sum_{i=1}^{K}\left(\frac{n_i\hat{p}_i-n_i\hat{p}_i^0}{\sqrt{n_i\hat{p}_i^0\hat{q}_i^0}}\right)^2=\sum_{i=1}^{K}\left(\frac{X_{i\cdot}-n_i\hat{p}_i^0}{\sqrt{n_i\hat{p}_i^0\hat{q}_i^0}}\right)^2$$

(c) 귀무가설 하에서 $p_1=\cdots=p_K$ 이므로, 공통의 모수를 p_0로 하고

$$z_i=\sqrt{n_i}\,(\hat{p}_i-p_0)/\sqrt{p_0q_0},\quad q_0=1-p_0,\quad z=(z_1\cdots z_K)^t$$

↳ 교재는 p^0, q^0를 사용.

라고 하면 귀무가설 하에서 $z\approx N_K(0,I_K)$

한편 귀무가설 하에서 $\hat{p}_i^0\simeq p_0$ 이므로

$$\sum_{i=1}^{K}\left(\frac{X_{i\cdot}-n_i\hat{p}_i^0}{\sqrt{n\hat{p}_i^0\hat{q}_i^0}}\right)^2=\sum_{i=1}^{K}n_i(\hat{p}_i-\hat{p}_i^0)^2/\hat{p}_i^0\hat{q}_i^0\simeq\sum_{i=1}^{K}n_i(\hat{p}_i-\hat{p}_i^0)^2/p_0q_0$$

그런데 $\hat{p}^0=\sum_{i=1}^{K}\frac{n_i\hat{p}_i}{n}$ 이므로

$$\sum_{i=1}^{K}n_i(\hat{p}_i-\hat{p}_i^0)^2=\sum_{i=1}^{K}n_i(\hat{p}_i-p_0+p_0-\hat{p}_i^0)^2$$

$$=\sum_{i=1}^{K}n_i(\hat{p}_i-p_0)^2+2n_i(\hat{p}_i-p_0)(p_0-\hat{p}_i^0)+n_i(p_0-\hat{p}_i^0)^2$$

$$=\sum n_i(\hat{p}_i-p_0)^2+2n(\hat{p}_i^0-p_0)(p_0-\hat{p}_i^0)+n(p_0-\hat{p}_i^0)^2$$

$$=\sum n_i(\hat{p}_i-p_0)^2-2n(p_0-\hat{p}_i^0)^2+n(p_0-\hat{p}_i^0)^2$$

$$=\sum n_i(\hat{p}_i-p_0)^2-n(p_0-\hat{p}_i^0)^2$$

따라서 $w=(\sqrt{n_1},\cdots,\sqrt{n_K})^t$ 라고 하면

$$\sum_{i=1}^{K}n_i(\hat{p}_i-\hat{p}_i^0)^2/p_0q_0=z^Tz-(w^tz)^2/(w^tw)=z^t(I-w(w^tw)^{-1}w^t)z$$

$$*\ w^Tz=\sum_{i=1}^{K}\sqrt{n_i}\cdot\sqrt{n_i}\,(\hat{p}_i-p_0)=n\hat{p}_i^0-np_0=n(\hat{p}_i^0-p_0)$$

$$\therefore\ (w^Tz)^2=n^2(\hat{p}_i^0-p_0),\quad (w^Tz)^2(w^Tw)^{-1}=n^2(\hat{p}_i^0-p_0)\cdot\frac{1}{n}=n(\hat{p}_i^0-p_0)$$

그러므로 귀무가설 $H_0:\ p_1=\cdots=p_K$ 하에서

$$2\left\{\ell(\hat{p})-\ell(\hat{p}^0)\right\}\simeq\sum_{i=1}^{K}\left(\frac{X_{i\cdot}-n_i\hat{p}_i^0}{\sqrt{n_i\hat{p}_i^0\hat{q}_i^0}}\right)^2\simeq\sum_{i=1}^{K}n_i(\hat{p}_i-\hat{p}_i^0)^2/p_0q_0\approx\chi^2(K-1)$$

Key takeaways

① Approximate $\ell(\hat{\theta}) - \ell(\hat{\theta}^0)$ using $t\log t \simeq (t-1) + \frac{1}{2}(t-1)^2$

② Find Z_i $\left(= \sqrt{n_i}(\hat{p}_i^0 - p_0)/\sqrt{p_0 q_0} \right) \sim N(0,1)$

③ denominator $\hat{p}_i^0, \hat{q}_i^0 \longrightarrow p_0, q_0$

④ Expand $\sum n_i(\hat{p}_i - \hat{p}_i^0) = \sum n_i(\hat{p}_i - p_0)^2 - n(\hat{p}_i^0 - p_0)^2$

* use the property: $\sum n_i \hat{p}_i = n\hat{p}_i^0$

⑤ Express it in the form $Z^T Z - (w^t z)^2 (w^t w)^{-1}$

* $w = (\sqrt{n_1}, \cdots, \sqrt{n_K})^T$

Q17) $K\,(K \geq 3)$개의 포아송 분포 $\text{Poisson}(\theta_i)$, $\theta_i \geq 0$ $(i = 1, \cdots, K)$ 에서의 서로 독립인 랜덤표본 $X_{i1}, \cdots, X_{in_i}$ $(i = 1, \cdots, K)$을 이용하여 다음 가설을 검정하려고 할 때 다음에 답하여라.

$$H_0: \theta_1 = \cdots = \theta_K \quad \text{vs} \quad H_1: \theta_1, \cdots, \theta_K \text{가 모두 같지는 않다.}$$

(a) 최대가능도비 검정통계량이 다음과 같이 주어지는 것을 밝혀라.

$$2\left\{\ell(\hat{\theta}) - \ell(\hat{\theta}^0)\right\} = 2\sum_{i=1}^{K} n_i \hat{\theta}_i^0 \left\{-\hat{\theta}_i/\hat{\theta}_i^0 + 1 + \hat{\theta}_i/\hat{\theta}^0 \log(\hat{\theta}_i/\hat{\theta}_i^0)\right\}$$

$$\hat{\theta}_i = \sum_{j=1}^{n_i} X_{ij}/n_i, \quad \hat{\theta}_i^0 = \sum_{i=1}^{K}\sum_{j=1}^{n_i} X_{ij}/n, \quad n = n_1 + \cdots + n_K$$

(b) 표본크기에 대하여

$n_i \longrightarrow \infty$, $n_i/(n_1 + \cdots + n_K) \longrightarrow \gamma_i$ $(0 < \gamma_i < 1)$ $(i = 1, \cdots, K)$

이 만족된다고 할 때, 귀무가설 H_0 하에서 최대가능도 검정통계량의 다음과 같은 근사에 대하여 설명하여라.

$$2\left\{\ell(\hat{\theta}) - \ell(\hat{\theta}^0)\right\} \simeq \sum_{i=1}^{K} \left(\frac{X_{i\cdot} - n_i\hat{\theta}_i^0}{\sqrt{n_i\hat{\theta}_i^0}}\right)^2, \quad X_{i\cdot} = \sum_{j=1}^{n_i} X_{ij}$$

(c) (b)로부터 최대가능도비 검정통계량이 귀무가설 H_0 하에서 근사적으로 $\chi^2(K-1)$ 분포를 따르는 것을 설명하여라.