8.1. 베르누이 분포 Bernoulli(p), $0 \le p \le 1$ 에서의 랜덤표본 $X_1, \cdots, X_n$을 이용한 두 추정량

$$\hat{p}^{(0)} = \sum_{i=1}^{n} X_i/n \quad , \quad \hat{p}^{(1)} = \frac{\sqrt{n}}{1+\sqrt{n}}\hat{p}^{(0)} + \frac{1}{1+\sqrt{n}} \cdot \frac{1}{2}$$

을 비교하려고 한다.

(a) 최대평균제곱오차 $\max_{0\le p\le 1} E_p[(\hat{p}-p)^2]$를 기준으로 두 추정량을 비교하여라.

(b) 다음의 베이지안 평균제곱오차를 기준으로 두 추정량을 비교하여라.

$$\int_0^1 E_p[(\hat{p}-p)^2]\, p^{\sqrt{n}-1}(1-p)^{\sqrt{n}-1}\, dp$$

(c) 평균제곱오차 $E_p[(\hat{p}-p)^2]$를 기준으로 $\hat{p}^{(1)}$이 $\hat{p}^{(0)}$보다 좋기 위한 p의 범위를 구하여라.

(Solution)

(a) $\hat{p}^{(0)} = \bar{x}$, $E[\hat{p}^{(0)}] = p$, $Var[\hat{p}^{(0)}] = p\cdot q/n$, $q = (1-p)$

$E_p[(\hat{p}^{(0)}-p)^2] = Var(\hat{p}^{(0)}) = p\cdot(1-p)\cdot n^{-1}$

$\max_{0\le p\le 1} E_p[(\hat{p}^{(0)}-p)^2] = \max_{0\le p\le 1} p\cdot(1-p)\cdot n^{-1} = \frac{1}{4n}$ $(\because p = 0.5)$

$$\begin{aligned} E_p[(\hat{p}^{(1)}-p)^2] &= E_p[(\hat{p}^{(1)} - E[\hat{p}^{(1)}] + E[\hat{p}^{(1)}] - p)^2] \\ &= E_p[(\hat{p}^{(1)} - E[\hat{p}^{(1)}])^2] + E_p[(E[\hat{p}^{(1)}]-p)^2] \\ &= Var(\hat{p}^{(1)}) + E_p\left[\left(\frac{\sqrt{n}}{1+\sqrt{n}}p + \frac{1}{1+\sqrt{n}}\cdot\frac{1}{2} - p\right)^2\right] \\ &= \frac{n}{(1+\sqrt{n})^2} Var(\hat{p}^{(0)}) + E_p\left[\left(\frac{1}{2(1+\sqrt{n})} - \frac{p}{1+\sqrt{n}}\right)^2\right] \\ &= \frac{1}{(1+\sqrt{n})^2} p(1-p) + \frac{(1-2p)^2}{4(1+\sqrt{n})^2} = \frac{4p-4p^2+1-4p+4p^2}{4(1+\sqrt{n})^2} = \frac{1}{4(1+\sqrt{n})^2} \end{aligned}$$

$\max_{0\le p\le 1} E_p[(\hat{p}^{(1)}-p)^2] = \max_{0\le p\le 1} \frac{1}{4(1+\sqrt{n})^2} = \frac{1}{4(1+\sqrt{n})^2}$

$\hat{p}^{(1)}$ approaches to p faster than $\hat{p}^{(0)}$ as sample size increases.

(b) $\int_0^1 E_p[(\hat{p}-p)^2]\, p^{\sqrt{n}-1}(1-p)^{\sqrt{n}-1}\, dp$

$\frac{\Gamma(\sqrt{n}+\sqrt{n})}{\Gamma(\sqrt{n})\Gamma(\sqrt{n})} p^{\sqrt{n}-1}(1-p)^{\sqrt{n}-1}$: pdf of Beta$(\sqrt{n},\sqrt{n})$

$$\Rightarrow \frac{\Gamma(\sqrt{n})\Gamma(\sqrt{n})}{\Gamma(2\sqrt{n})}\int_0^1 Var(p)\cdot\frac{\Gamma(2\sqrt{n})}{\Gamma(\sqrt{n})\Gamma(\sqrt{n})} p^{\sqrt{n}-1}(1-p)^{\sqrt{n}-1}\, dp$$

① $\int_0^1 E_p[(\hat{p}^{(0)}-p)^2]\, p^{\sqrt{n}-1}(1-p)^{\sqrt{n}-1}\, dp = \int_0^1 \frac{1}{n} p^{\sqrt{n}}(1-p)^{\sqrt{n}}\, dp$

$= \frac{1}{n}\cdot\frac{\Gamma(\sqrt{n}+1)^2}{\Gamma(2\sqrt{n}+2)}\cdot\int_0^1 \frac{\Gamma(2\sqrt{n}+2)}{\Gamma(\sqrt{n}+1)^2}\cdot p^{\sqrt{n}}(1-p)^{\sqrt{n}}\, dp = \underline{\frac{1}{n}\frac{\Gamma(\sqrt{n}+1)^2}{\Gamma(2\sqrt{n}+2)}}$ ✓✓

*

② $\int_0^1 E_p[(\hat{p}^{(1)}-p)^2]\, p^{\sqrt{n}-1}(1-p)^{\sqrt{n}-1}\, dp$

$= \int_0^1 \frac{1}{4(1+\sqrt{n})^2} p^{\sqrt{n}-1}(1-p)^{\sqrt{n}-1}\, dp = \frac{1}{4(1+\sqrt{n})^2}\frac{\Gamma(\sqrt{n})^2}{\Gamma(2\sqrt{n})}$ ✓✓

* $\frac{1}{n}\frac{\Gamma(\sqrt{n}+1)^2}{\Gamma(2\sqrt{n}+2)} = \frac{\cancel{\sqrt{n}\cdot\sqrt{n}}\cdot\Gamma(\sqrt{n})^2}{\cancel{n}\cdot(2\sqrt{n}+1)\cdot 2\sqrt{n}\cdot\Gamma(2\sqrt{n})} = \frac{\Gamma(\sqrt{n})^2}{4(n+\sqrt{n})\Gamma(2\sqrt{n})} > \frac{\Gamma(\sqrt{n})^2}{4(n+2\sqrt{n}+1)\Gamma(2\sqrt{n})}$

$= \frac{\Gamma(\sqrt{n})^2}{4(\sqrt{n}+1)^2\Gamma(2\sqrt{n})}$

$\therefore \int_0^1 E_p[(\hat{p}^{(1)}-p)^2]\, p^{\sqrt{n}-1}(1-p)^{\sqrt{n}-1}\, dp < \int_0^1 E_p[(\hat{p}^{(0)}-p)^2]\, p^{\sqrt{n}-1}(1-p)^{\sqrt{n}-1}\, dp$

(c) $\frac{1}{4(1+\sqrt{n})^2} \le \frac{p(1-p)}{n}$

$\Rightarrow \frac{n}{4(1+\sqrt{n})^2} = C \le p(1-p)$

if $n=100$, $\frac{100}{4(11)^2} = C \le \frac{100}{121\cdot 9} = \frac{\cancel{100}}{\cancel{484}\ 121} = \frac{25}{121}$

$-p^2+p-C \ge 0$

$p^2-p+C \le 0$

$p^2-p+1/4-1/4+C \le 0$

$(p-1/2)^2 - 1/4 + C \le 0$

$(p-1/2)^2 \le 1/4 - C$

$p-1/2 \le (1/4-C)^{1/2}$ or $\frac{1}{2}-p \le (1/4-C)^{1/2} \Leftrightarrow p \ge \frac{1}{2} - (1/4-C)^{1/2}$

$\therefore \frac{1}{2} - (1/4-C)^{1/2} \le p \le \frac{1}{2} + (1/4-C)^{1/2}$ ✓✓

$\frac{1}{4} - C = \frac{1}{4} - \frac{n}{4(1+\sqrt{n})^2} = \frac{1+2\sqrt{n}+n-n}{4(1+\sqrt{n})^2} = \frac{1+2\sqrt{n}}{4(1+\sqrt{n})^2}$

↑ took 1 hour 30min to solve. need to solve it faster.

8.2. 다음 각 모형에서의 랜덤표본을 $X_1, \cdots, X_n$이라고 할 때, $\theta \in \Omega$에 관한 충분통계량을 구하여라.

(a) Beta$(1,\theta)$, $\theta \in \Omega$, $\Omega = (0, +\infty)$

(b) Beta(θ,θ), $\theta \in \Omega$, $\Omega = (0, +\infty)$

(a) $f(x;\theta) = \theta(1-x)^{\theta-1} = exp\{\log\theta + (\theta-1)\log(1-x_i)\}$

The Beta distribution is an exponential family and satisfies the necessary conditions for $\Sigma \log(1-X_i) = Y$ to be a complete sufficient statistic for θ.

(i) Support of the distribution $\mathcal{X} := \{x : 0 < x < 1\}$ does not depend on θ.

(ii) parameter space contains an open box $(a_1,b_1)\times(a_2,b_2) \in \mathbb{R}^2$

(iii) $Var[T(X)] \ne 0$

(b) $f(x;\theta) = \frac{\Gamma(\theta+\theta)}{\Gamma(\theta)\Gamma(\theta)} x^{\theta-1}(1-x)^{\theta-1} I_{(0,1)}(x)$

the pdf does not satisfy condition (ii) above.

$$\prod_{i=1}^n f(x_i;\theta) = \left[\frac{\Gamma(\theta+\theta)}{\Gamma(\theta)\Gamma(\theta)}\left(\prod_1^n x_i\right)^{\theta-1}\left(\prod_1^n (1-x_i)\right)^{\theta-1}\right] \cdot I_{(0<x_{(1)},\, x_{(n)}<1)}$$

$$= \left[\frac{\Gamma(2\theta)}{\Gamma(\theta)^2}\prod_{i=1}^n (x_i(1-x_i))^{\theta-1}\right] I_{(0<x_{(1)},\, x_{(n)}<1)}$$

$\therefore$ By the factorization theorem, $\prod_{i=1}^n x_i(1-x_i)$ is a SS

★ 8.3. 다변량 정규분포 $N_k(\mu,\Sigma)$, $\mu\in\mathbb{R}^k$, $\det(\Sigma)>0$ 에서의 랜덤표본 $X_1,\cdots,X_n$ 이라고 할 때 (μ,Σ)에 관한 충분통계량을 구하여라.

$$f(x;\theta)=|\det(2\pi\Sigma)|^{-1/2}\exp\left\{-\tfrac{1}{2}(x-\mu)^t\Sigma^{-1}(x-\mu)\right\}$$

$$=|\det(2\pi\Sigma)|^{-1/2}\exp\left\{-\tfrac{1}{2}\left(x^t\Sigma^{-1}x-2\mu^t\Sigma^{-1}x+\mu^t\mu\right)\right\}$$

$\Sigma=P\Lambda P^t$, $\Sigma^{-1}=P\Lambda^{-1}P^t$, $\Lambda=\mathrm{diag}(\lambda_1,\cdots,\lambda_k)$

$$\prod_{i=1}^n f(x_i;\theta)=|\det(2\pi\Sigma)|^{-n/2}\left\{-\tfrac{1}{2}\left(\sum_{i=1}^n x_i^t\Sigma^{-1}x_i-2\mu^t\Sigma^{-1}\sum_{i=1}^n x_i+n\cdot\mu^t\mu\right)\right\}$$

$x^t\Sigma^{-1}x=x^tP\Lambda^{-1}P^tx=(\Lambda^{-1/2}P^tx)^t(\Lambda^{-1/2}P^tx)$

$$=(\det(2\pi\Sigma))^{-n/2}\exp\left\{-\tfrac{1}{2}\mathrm{trace}\left(\Sigma^{-1}\sum_{i=1}^n x_ix_i^t\right)+\mu^t\Sigma^{-1}\sum_{i=1}^n x_i+\tfrac{n}{2}\mu^t\mu\right\}$$

$\therefore\left(\sum_{i=1}^n X_i,\sum_{i=1}^n X_iX_i^t\right)$ is the SS by the factorization theorem.

and its one-to-one transformation $\left(\bar{X},\sum_{i=1}^n(X_i-\bar{X})(X_i-\bar{X})^t\right)$ is also a sufficient statistic of $\theta=(\mu,\Sigma)^t$

8.4. 다음 각 모형에서의 랜덤표본을 $X_1,\cdots,X_n$ 이라고 할 때, $\theta=(\mu,\sigma)^t\in\Omega$ 에 관한 충분통계량을 구하여라.

(a) 로지스틱분포 $L(\mu,\sigma)$, $-\infty<\mu<+\infty$, $\sigma>0$

(b) 코쉬분포 $C(\mu,\sigma)$, $-\infty<\mu<+\infty$, $\sigma>0$

(a) $$f(x;\mu,\sigma)=\frac{e^{-(x-\mu)/\sigma}}{\sigma(1-e^{-(x-\mu)/\sigma})^2}=\frac{e^{(x-\mu)/\sigma}}{\sigma(1-e^{(x-\mu)/\sigma})^2}=\frac{1}{4\sigma}\cdot sech^2\left(\frac{x-\mu}{\sigma}\right)$$

* $sech(x)=\dfrac{1}{\cosh(x)}=\dfrac{e^x+e^{-x}}{2}$ $\therefore sech(x)=\dfrac{1}{e^x+e^{-x}}$

derivation: $1+e^{-t}=e^{-t/2}(e^{t/2}+e^{-t/2})$, $(1+e^{-t})^2=e^{-t}(e^{t/2}+e^{-t/2})^2$ … (생략)

$$\prod_{i=1}^n f(x_i;\theta)=\prod_{i=1}^n\frac{e^{(x-\mu)/\sigma}}{\sigma\cdot(1-e^{(x-\mu)/\sigma})^2}=\prod_{i=1}^n\frac{e^{(x-\mu)/\sigma}}{\sigma^2(1-2\cdot e^{(x-\mu)/\sigma}+e^{2(x-\mu)/\sigma})}$$

$$=\sigma^{-n}\exp\left\{\sum_{i=1}^n x_{(i)}-n\mu\right\}\prod_{i=1}^n\left[1+\exp\left(\frac{x_{(i)}-\mu}{\sigma}\right)\right]$$

$$=ft^n(\mu,\sigma,x_{(1)},\cdots,x_{(n)})$$

From the factorization theorem, $(X_{(1)},\cdots,X_{(n)})^t$ is a SS for $\theta=(\mu,\sigma)^t\in\Omega$

(b) $\Omega=\{(\mu,\sigma)^t:-\infty<\mu<+\infty,\ \sigma>0\}$

$$pdf(x;\mu,\sigma)=\prod_{i=1}^n\frac{1}{\sigma}\frac{1}{1+\{(x_i-\mu)/\sigma\}^2}$$

$$=\sigma^{-n}\prod_{i=1}^n\left[1+\left(\frac{x_{(i)}-\mu}{\sigma}\right)^2\right]^{-1}$$

$$=ft^n(\mu,\sigma,x_{(1)},\cdots,x_{(n)})$$

$\therefore$ From the factorization theorem, $(X_{(1)},\cdots,X_{(n)})^t$ is a SS for $\theta=(\mu,\sigma)^t\in\Omega$

8.5. 실수 전체의 집합에서의 연속형 분포를 모형으로 하는 경우에는 모수공간을

$$\Omega=\{f:f\text{는 수직선 위의 연속형 확률밀도함수}\}$$

로 나타낼 수 있다. 이러한 모형에서의 랜덤표본을 $X_1,\cdots,X_n$ 이라고 할 때 $f\in\Omega$ 에 관한 충분통계량을 구하여라.

$$pdf(x;f)=\prod_{i=1}^n f(x_i)=\prod_{i=1}^n f(x_{(i)})=ft^n(f,x_{(1)},\cdots,x_{(n)})\times 1$$

$\therefore$ By the factorization theorem, $(X_{(1)},\cdots,X_{(n)})^t$ is a SS for $f\in\Omega$

★ 8.6. 자연수 전체의 집합에서의 이산형 분포를 모형으로 하는 경우에는 모수공간을

$$\Omega=\left\{(p_1,p_2,\cdots):p_K\ge 0,\ K=1,2,\cdots,\ \sum_{K=1}^{\infty}p_K=1\right\}$$

로 나타낼 수 있다. 이러한 모형에서의 랜덤표본을 $X_1,\cdots,X_n$ 이라고 할 때 $(p_1,p_2,\cdots)\in\Omega$ 에 관한 충분통계량을 구하여라.

모수를 $p=(p_1,p_2,\cdots)$로 나타내고, R을 $X_1,\cdots,X_n$ 중에서 서로 다른 것의 개수, $X_{(1)}<\cdots<X_{(R)}$ 을 이들을 크기 순서로 늘어놓은 순서통계량, $N_{(1)},\cdots,N_{(R)}$을 이들의 반복회수라고 하면 다음과 같은 분해식이 가능하다.

$$pdf(x;p)=\prod_{i=1}^n p_{x_i}=[p_{x_{(1)}}]^{n_{(1)}}\cdots[p_{x_{(r)}}]^{n_{(r)}}=ft^n(p,r,(x_{(1)},n_{(1)}),\cdots,(x_{(r)},n_{(r)}))\times 1$$

따라서 분해정리로부터 $(R,X_{(1)},\cdots,X_{(r)},N_{(1)},\cdots,N_{(R)})^t$이 $p\in\Omega$에 관한 충분통계량이다.

★ 8.7. 두 개의 모수를 갖는 지수분포 $Exp(\mu,\sigma)$, $-\infty<\mu<+\infty$, $\sigma>0$ 에서의 랜덤표본을 관측할 수 없고 이들의 순서통계량 중에서 $X_{(1)}<\cdots<X_{(r)}$ $(1\le r<n)$ 만을 관측할 수 있다고 한다. 이 때 $(\mu,\sigma)^t\in\mathbb{R}\times\mathbb{R}_+$에 관한 충분통계량을 구하여라.

$$f(x;\theta)=\frac{1}{\sigma}\exp\left\{-\frac{x-\mu}{\sigma}\right\}\quad \text{let } t=\frac{x-\mu}{\sigma},\ f(t;\theta)=e^{-t},\ T\sim Exp(1)$$

$$P(T<t)=1-e^{-t},\ P(T>t)=1-P(T<t)=e^{-t}$$

$$\prod_{i=1}^n f(x_i;\theta)\Rightarrow\prod_{i=1}^r f(x_{(i)};\theta)\cdot\prod_{i=r+1}^n(1-F_X(x_{(r)}))\cdot\frac{n!}{1!\cdots1!(n-r)!}$$

$$=\prod_{i=1}^r\frac{1}{\sigma}\exp\left\{-\frac{x_{(i)}-\mu}{\sigma}\right\}\cdot\prod_{i=r+1}^n\exp\left\{-\frac{x_{(r)}-\mu}{\sigma}\right\}I_{(\mu,\infty)}(x_{(1)})$$

$$=\sigma^{-r}\exp\left\{-\frac{1}{\sigma}\left(\sum_{i=1}^r x_{(i)}-r\mu\right)\right\}\cdot\exp\left\{-\frac{1}{\sigma}(n-r)(x_{(r)}-\mu)\right\}I_{(\mu,\infty)}(x_{(1)})$$

$$=\sigma^{-r}\exp\left\{-\frac{1}{\sigma}\left(\sum_{i=1}^r x_{(i)}+(n-r)x_{(r)}-\mu\right)\right\}I_{(\mu,\infty)}(x_{(1)})$$

$\therefore$ By the factorization theorem, $\left(\sum_{i=1}^r x_{(i)},x_{(r)}\right)^t$ are SS for $\theta=(\mu,\sigma)^t\in\Omega$

$\left(\sum_{i=1}^r x_{(i)}-(n-r)x_{(r)},x_{(1)}\right)^t$

8.8. 균등분포 $U[-\theta,\theta]$, $\theta>0$ 에서의 랜덤표본을 $X_1,\cdots,X_n$ 이라고 할 때 다음에 답하여라.

(a) $Y=\max_{1\le i\le n}|X_i|$ 가 $\theta\in(0,+\infty)$ 에 관한 충분통계량임을 밝혀라.

(b) 랜덤표본의 순서통계량을 $X_{(1)}<\cdots<X_{(n)}$ $(n\ge 2)$ 이라고 할 때,

$$\hat{\theta}=C_n(X_{(n)}-X_{(1)})$$

이 θ의 불편추정이 되기 위한 C_n의 값을 구하여라.

(c) (a)에서의 충분통계량 $Y=\max_{1\le i\le n}|X_i|$ $(n\ge 2)$ 에 기초한 추정량으로서 (b)에서의 불편추정량보다 더 작은 평균제곱오차를 갖는 추정량을 구하여라.

(a) $f(x;\theta)=\frac{1}{2\theta}I_{[-\theta,\theta]}(|x_i|)$

$$\prod_{i=1}^n f(x_i;\theta)=(2\theta)^{-n}I_{[-\theta<x_{(1)},x_{(n)}<+\theta]}=(2\theta)^{-n}I_{[\max(-x_{(1)},x_{(n)})<\theta]}$$

$\therefore$ By the factorization theorem, $\max(-x_{(1)},x_{(n)})$ is a SS for θ.
Also, the 1-1 transformation $\max_{1\le i\le n}|X_i|$ is a SS for θ.

(b) $E[C_n(X_{(n)}-X_{(1)})]=\theta$

$X\sim U[-\theta,\theta]\Rightarrow X+\theta\sim U[0,2\theta]\Rightarrow \frac{X+\theta}{2\theta}\sim U[0,1]$

$\therefore \left(\frac{X+\theta}{2\theta}\right)_{(1)}\sim Beta(1,n)$, $\left(\frac{X+\theta}{2\theta}\right)_{(n)}\sim Beta(n,1)$

$$E\left[\frac{X_{(1)}+\theta}{2\theta}\right]=\frac{1}{1+n}\iff E[X_{(1)}]=\frac{2\theta}{1+n}-\theta$$

$$E\left[\frac{X_{(n)}+\theta}{2\theta}\right]=\frac{n}{n+1}\iff E[X_{(n)}]=\frac{2\theta n}{1+n}-\theta$$

$$E[X_{(n)}-X_{(1)}]=E[X_{(n)}]-E[X_{(1)}]=\frac{2\theta n}{1+n}-\frac{2\theta}{1+n}=2\theta\cdot\frac{n-1}{1+n}$$

$\therefore C_n=\frac{n+1}{2(n-1)}$

(c) $E[Y]=E[\max_{1\le i\le n}|X_i|]=\frac{1}{2}X_{(n)}-\frac{1}{2}X_{(1)}=\left(\frac{2\theta n}{1+n}-\theta-\frac{2\theta}{1+n}+\theta\right)\frac{1}{2}$

$$=\left(\frac{2\theta(n-1)}{1+n}\right)\cdot\frac{1}{2}=\theta\frac{n-1}{n+1}$$

$\therefore E\left[\frac{n+1}{n-1}Y\right]=\theta$

$\therefore \hat{\theta}_{UMVUE}=\frac{n+1}{n-1}Y$

* why this doesn't work:

$$\max_{1\le i\le n}|X_i|\neq\begin{cases}-X_{(1)} & \text{with probability } 1/2\\ X_{(n)} & \text{with probability } 1/2\end{cases}$$

$\because M=\max\{-X_{(1)},X_{(n)}\}$ is a deterministic function of Both order statistics Simultaneously.

Also Expectation is linear only for linear combinations, not for nonlinear operations like max.

In fact, $E[\max(A,B)]=\int P(\max(A,B)>t)\,dt$ which depends on the joint distribution of A and B.
It is true that $X_{(1)}\overset{d}{=}-X_{(n)}$ because the distribution is symmetric. But symmetry only implies $E[X_{(1)}]=-E[X_{(n)}]$
It does not imply that the larger of the two occurs with probability 1/2.

The probability that the maximum value comes from the positive side is: $P(X_{(n)}>-X_{(1)})$
This is not equal to 1/2 because if the sample happens to contain an extreme value, it pushes the entire distribution of order statistics. ⇒ The two extremes are dependent.

Another reason: $X_{(1)}$ and $X_{(n)}$ are strongly dependent.

Example: A=1, B=100

$\max(A,B)=100$ But: $0.5A+0.5B=50.5$

ChatGPT Approach:

$M=\max\{-X_{(1)},X_{(n)}\}$

Let $Y_i=|X_i|$. Then, $Y_i\sim U[0,\theta]$ $\therefore M=\max(Y_1,\cdots,Y_n)$

$P(Y_{(n)}\le y)=\frac{n!}{\theta^n}y^{n-1}$, $E[M]=E[Y_{(n)}]=\int_0^\theta y\cdot\frac{ny^{n-1}}{\theta^n}dy=\frac{n}{n+1}\theta$

$\therefore E\left[\frac{n+1}{n}M\right]=\theta$ $\therefore \hat{\theta}_{UMVUE}=\frac{n+1}{n}\max_{1\le i\le n}|X_i|$

☆ 8.9. 기하분포 $Geo(p)$, $0<p<1$ 에서의 랜덤표본을 $X_1,\cdots,X_n$ $(n\ge 2)$ 이라고 할 때, $p\in(0,1)$ 에 관한 완비충분통계량과 p의 전역최소분산불편추정량을 구하여라.

$$f(x;p)=(1-p)^{x-1}p,\quad \prod_{i=1}^n f(x_i;p)=(1-p)^{\sum x_i-n}p^n$$

$$=\exp\{(x-1)\log(1-p)+\log p\}$$

(i) Support $\mathcal{X}:=\{x: x=1,2,3,\cdots\}$
(ii) $\Omega=(0,1)$
(iii) $Var(T(X))\neq 0$

$\therefore \sum x_i$ is a CSS for p

$\bar{x}$ is also CSS for p ($\because$ 1-1 transformation)

$$M_X(t)=E[e^{tX}]=\sum_{x=1}^\infty e^{tx}(1-p)^{x-1}p=\sum_{x=1}^\infty e^{tx}q^{x-1}p=\sum_{x=1}^\infty(qe^t)^{x-1}\cdot(1-qe^t)\cdot\frac{e^tp}{1-qe^t}$$

$$C_X(t)=\log(e^t)+\log(p)-\log(1-qe^t)=\frac{e^tp}{1-qe^t}$$
$$=t+\log(p)-\log(1-qe^t)$$

$$C'_X(t)=1+\frac{qe^t}{1-qe^t},\quad C'_X(t=0)=1+\frac{q}{p}=\frac{p+q}{p}=\frac{1}{p}$$

$$C_X''(t)=\frac{qe^t}{1-qe^t}+\frac{(qe^t)^2}{(1-qe^t)^2}=\frac{qe^t-(qe^t)^2+(qe^t)^2}{(1-qe^t)^2}=\frac{qe^t}{(1-qe^t)^2}$$

$$\therefore Var(X)=C_X''(t=0)=\frac{q}{p^2}$$

$\bar{x}\xrightarrow[n\to\infty]{d}N(p^{-1},\frac{q}{p^2n^1})$ $\because$ CLT

r: # of success
y: # of trials until r^{th} success

$\sum_{i=1}^r x_i\sim NB(r,p)$. Let $Y=\sum_{i=1}^n x_i$. $f_Y(y)=\binom{y}{r-1}(1-p)^{y-r}p^r$

$$E[Y]=\sum_{y=r}^\infty y\binom{y}{r-1}\cdot(1-p)^{y-r}\cdot p^r=\frac{r}{p}$$

$E[g(Y)] = p \Leftrightarrow \sum_{y=r}^{\infty} g(y) \cdot \binom{y-1}{r-1} \cdot (1-p)^{y-r} \cdot p^r = p$

$\Leftrightarrow \sum_{y=r}^{\infty} g(y) \cdot \frac{(y-1)!}{(r-1)!(y-r+1)!} (1-p)^{y-r} p^{r-1} = 1$

$\Leftrightarrow \sum_{y=r}^{\infty} \frac{(y-2)!}{(r-2)!(y-r+1)} (1-p)^{y-r} p^{r-1} = 1$

$\Leftrightarrow \sum_{y=r-1}^{\infty} \binom{y-2}{r-2} (1-p)^{y-r} p^{r-1} = 1$

$\therefore g(y) = \frac{r-1}{y-1} = \frac{r-1}{\sum x_i - 1} \quad r=n \quad \therefore \hat{p}^{UMVUE} = \frac{1}{\frac{1}{n-1}(\sum x_i - 1)} \quad (n \geq 2)$

8.10 확률밀도함수가 $f(x;\mu) = e^{-(x-\mu)} I_{[\mu,\infty)}(x)$ 로 주어지는 지수분포 $Exp(\mu, 1]$, $-\infty < \mu < +\infty$ 에서의 랜덤표본을 $X_1, \cdots, X_n$ 이라고 할 때, $\mu \in (-\infty, +\infty)$ 에 관한 완비충분통계량과 μ의 전역최소분산 불편추정량을 구하여라.

(i) The support of the distribution depends on the parameter μ.

$\prod_{i=1}^{n} f(x_i;\mu) = e^{-n(\bar{x}-\mu)} I_{[\mu,\infty)}(x_{(1)})$

By the factorization theorem, $x_{(1)}$ is a sufficient statistic

$X - \mu \sim Exp(1) \overset{d}{=} Gamma(1,1)$, $(x-\mu)_{(1)}$ pdf : $\frac{n!}{1!(n-1)!} e^{-x} \cdot (1-(1-e^{-x}))^{n-1}$

$= n e^{-nx}$

Let $Y' = x_{(1)} - \mu$, $E[g(Y')] = \int_0^{\infty} g(Y') \cdot n e^{-ny} dx = 0$

$\Rightarrow g(Y') \cdot n = 0$ ($\because$ uniqueness property of the Laplace transform)

$\therefore g(Y') = 0$

Let $h(Y) = g(Y')$ $\therefore E[h(Y)] = 0 \Leftrightarrow h(Y) = 0$

$\therefore x_{(1)}$ is a CSS for μ

$E[x_{(1)} - \mu] = \int_0^{\infty} y \cdot n e^{-ny} dy = \left[y \cdot -e^{-ny} \right]_0^{\infty} - \int_0^{\infty} 1 \cdot -e^{-ny} dy$

$= \int_0^{\infty} e^{-ny} dy = \left[-\frac{1}{n} e^{-ny} \right]_0^{\infty} = \frac{1}{n}$

$\therefore E[x_{(1)}] = \mu + \frac{1}{n}$ $\therefore \hat{\mu}^{UMVUE} = x_{(1)} - \frac{1}{n}$

[Textbook Approach]

From 〈Ex. 4.3.3〉, $n(x_{(1)} - \mu) \sim Exp(1)$

따라서 적분과 미분의 관계로부터 다음이 성립함을 알 수 있다.

$E_\mu g(x_{(1)}) \overset{\mu}{=} 0 \Leftrightarrow \int_{-\infty}^{+\infty} g(\frac{z}{n} + \mu) e^{-z} I_{(0,\infty)}(z) dz \overset{\mu}{=} 0$

$\Leftrightarrow \int_{\mu}^{\infty} g(x) e^{-n(x-\mu)} n dx \overset{\mu}{=} 0$

$\Rightarrow g(\mu) e^{-n\mu} \overset{\mu}{=} 0$

$(\cdots)$

$\hat{\mu}^{UMVUE} = x_{(1)} - \frac{1}{n}$

★ 8.11. 두 개의 모수를 갖는 지수분포 $Exp(\mu, \sigma)$, $-\infty < \mu < +\infty$, $\sigma > 0$ 에서의 랜덤표본을 $X_1, \cdots, X_n$ $(n \geq 2)$ 이라고 할 때 다음에 답하여라.

(a) $(\mu, \sigma)^t \in \mathbb{R} \times \mathbb{R}_+$ 에 관한 완비충분통계량을 구하여라.

(b) μ의 전역최소분산 불편추정량을 구하여라.

(c) σ의 전역최소분산 불편추정량을 구하여라.

$f(x;\theta) = \frac{1}{\sigma} \exp\left\{ \frac{x-\mu}{\sigma} \right\} I_{[\mu,\infty)}(x)$

$\prod_{i=1}^{n} f(x_i;\theta) = \sigma^{-n} \exp\left\{ \frac{1}{\sigma} (\sum x_i - n\mu) \right\} I_{[\mu,\infty)}(x_{(1)})$

$\therefore$ By the factorization theorem, $\sum X_i$ and $X_{(1)}$ are sufficient statistics.

we already proved in problem 8.10 that $X_{(1)}$ is a CSS

$(X-\mu)/\sigma \sim Exp(1)$, $\sum (\frac{x_i - \mu}{\sigma}) \overset{d}{=} Gamma(n,1)$

$\frac{1}{\sigma} \left\{ \sum x_i - n\mu \right\} \sim Gamma(n,1)$

Let $Y = \frac{1}{\sigma} \left\{ \sum x_i - n\mu \right\}$, $E[h(Y)] = \int_0^{\infty} h(y) \cdot \frac{1}{\Gamma(n)} y^{n-1} \exp\{-y\} dy \equiv 0$

$\Leftrightarrow h(y) \cdot y^{n-1} \equiv 0$ ($\because$ 라플라스 변환의 유일성)

$\Leftrightarrow h(y) \equiv 0$

$\therefore \sum x_i$ is a CSS.

$(X-\mu)/\sigma \sim Exp(1) \Rightarrow n(x_{(1)} - \mu)/\sigma \sim Exp(1)$

$\therefore E[\frac{n}{\sigma}(x_{(1)} - \mu)] = 1 \Leftrightarrow E[x_{(1)}] = \mu + \frac{\sigma}{n}$

$\sum (x - \mu)/\sigma \sim Gamma(n,1)$

$\therefore E[\sum (x_i - \mu)/\sigma] = n \Leftrightarrow E[\sum x_i] = n\sigma + n\mu = n(\sigma + \mu)$

$\Leftrightarrow E[\bar{x}] = \sigma + \mu$

$E[\bar{x} - x_{(1)}] = \frac{n-1}{n} \sigma$ $\therefore \hat{\sigma}^{UMVUE} = \frac{1}{n-1} \sum (x_i - x_{(1)})$ ✓✓

$E[x_{(1)}] - \frac{1}{n-1} E[\bar{x} - x_{(1)}] = \mu + \frac{\sigma}{n} - \frac{1}{n}\sigma = \mu$

$\therefore \hat{\mu}^{UMVUE} = (1 + \frac{1}{n-1}) x_{(1)} - \frac{1}{n-1} \bar{x} = \frac{n}{n-1} x_{(1)} - \frac{1}{n-1} \bar{x}$ ✓✓

[Textbook Approach]

(a) $(x_i - \mu)/\sigma \overset{iid}{\sim} Exp(1)$ $(i = 1, 2, \cdots)$

이므로 표준지수로부터의 순서통계량에 관한 〈예 4.3.3〉으로부터

$n(x_{(1)} - \mu)/\sigma$, $(n-r+1)(x_{(r)} - x_{(r-1)})/\sigma$ $(r = 2, \cdots, n) \overset{iid}{\sim} Exp(1)$

임을 알 수 있다. 따라서 $(x_{(1)}, \sum_1^n x_i)^t$ 의 일대일 함수로서 $(\mu, \sigma)^t \in \mathbb{R} \times \mathbb{R}_+$ 에 관한 충분통계량인

$(S,T)^t = (X_{(1)}, \sum_{i=1}^{n}(X_i - X_{(1)})^t = (X_{(1)}, \sum_{r=2}^{n}(n-r+1)(X_{(r)} - X_{(r-1)})^t$

을 생각하면 $n(S-\mu)/\sigma \sim Exp(1)$, $T/\sigma \sim Gamma(n-1,1)$

이고 S와 T는 서로 독립이다.

이를 이용하면 $(S,T)^t$ 가 완비통계량임을 다음과 같이 적분과 미분의 관계와 라플라스 변환의 유일성으로부터 밝힐 수 있다.

$$E_{\mu,\sigma}\, g(S,T) = \int_0^{+\infty}\int_0^{+\infty} g\left(\frac{w}{n}+\mu, w\sigma\right) e^{-z}\frac{1}{\Gamma(n-1)} w^{n-2} e^{-w}\, dz\,dw \overset{\mu,\sigma}{\equiv} 0$$

$$\Leftrightarrow \int_\mu^{\infty}\int_0^{+\infty} g(x,y)\, e^{-n(x-\mu)/\sigma} y^{n-2} e^{-y/\sigma}\, dy\,dx \overset{\mu,\sigma}{\equiv} 0$$

$$\Leftrightarrow \int_\mu^{\infty}\int_0^{\infty} g(x,y)\, e^{-nx/\sigma} y^{n-2} e^{-y/\sigma}\, dy\,dx \overset{\mu,\sigma}{\equiv} 0$$

$$\Leftrightarrow \int_0^{\infty} g(\mu,y)\, e^{-n\mu/\sigma} y^{n-2} e^{-y/\sigma}\, dy \overset{\mu,\sigma}{\equiv} 0 \quad (\because \text{적분과 미분의 관계})$$

$$\Rightarrow \int_0^{+\infty} g(\mu,y)\cdot y^{n-2} e^{-y/\sigma}\, dy \overset{\mu,\sigma}{\equiv} 0$$

$$\Rightarrow g(\mu,y)\, y^{n-2} \overset{\mu,y}{\equiv} 0, \text{ 즉 } g(x,y) \overset{x,y}{\equiv} 0 \quad (\because \text{라플라스 변환의 유일성})$$

따라서 $(S,T)^t = (X_{(1)}, \sum_i (X_i - X_{(1)})^t$ 는 $(\mu,\sigma)^t \in \mathbb{R}\times\mathbb{R}_+$ 에 관한 완비충분통계량이다.

8.12. 연습문제 (8.8)에서 다음에 답하여라.

(a) $S = (\min_{1\le i\le n} X_i, \max_{1\le i\le n} X_i)^t$ 가 $\theta\in(0,+\infty)$ 에 관한 완비통계량이 아님을 밝혀라.

(b) $Y = \max_{1\le i\le n}|X_i|$ 가 $\theta\in(0,+\infty)$ 에 관한 완비충분통계량임을 밝히고, θ의 전역최소분산 불편추정량을 구하여라.

(a) $X \sim U[-\theta,\theta]$, $X+\theta \sim U[0,2\theta]$, $(X+\theta)/2\theta \sim U[0,1]$

$(X_{(1)}+\theta)/2\theta \sim Beta(1,n)$, $(X_{(n)}+\theta)/2\theta \sim Beta(n,1)$

$$E[X_{(1)}] = \frac{1}{1+n}\cdot 2\theta - \theta = \frac{2-1-n}{1+n}\theta = \frac{1-n}{1+n}\theta$$

$$E[X_{(n)}] = \frac{n}{1+n}\cdot 2\theta - \theta = \frac{2n-1-n}{1+n}\theta = \frac{n-1}{1+n}\theta$$

$\therefore E[X_{(1)} + X_{(n)}] = 0$

(b) $Y = \max_{1\le i\le n}|X_i|$ $\quad X\sim U(-\theta,\theta) \Rightarrow |X| \sim U(0,\theta)$

$$pdf_Y(y) = \frac{n!}{1!\,(n-1)!} F_{|X|}(y)\cdot f_{|X|}(y) = n\cdot\left(\frac{y}{\theta}\right)^{n-1}\frac{1}{\theta} = n\theta^{-n} y^{n-1}$$

$$E[g(Y)] = \int_0^{\theta} g(y)\cdot n\theta^{-n}\cdot y^{n-1}\, dy \equiv 0 \Leftrightarrow \int_0^{\theta} g(y)\cdot y^{n-1}\, dy \equiv 0$$

$$\Leftrightarrow g(\theta)\theta^{n-1} \equiv 0$$

$$\therefore g(y) = 0$$

$|X_i| \sim U(0,\theta) \Rightarrow |X|/\theta \sim U(0,1) \Rightarrow \max_{1\le i\le n}|X_i|/\theta \sim Beta(n,1)$

$$\therefore E\left[\max_{1\le i\le n}|X_i|/\theta\right] = \frac{n}{1+n}$$

$$\therefore \hat{\theta}^{UMVUE} = \frac{1+n}{n}\max_{1\le i\le n}|X_i| \quad \checkmark\checkmark$$

8.13. 균등분포 $U[-\theta,2\theta]$, $\theta>0$ 에서 랜덤표본을 $X_1,\cdots,X_n$ 이라고 할 때 다음에 답하여라.

(a) $\theta\in(0,+\infty)$ 에 관한 최소충분통계량을 구하여라.

(b) θ의 전역최소분산불편추정량을 구하여라.

$X\sim U[-\theta,2\theta]$, $X+\theta\sim U[0,3\theta]$, $(X+\theta)/3\theta \sim U[0,1]$

$$f(x;\theta) = \frac{1}{3\theta} I_{[-\theta,2\theta]}(x), \quad \prod_{i=1}^{n} f(x_i;\theta) = \frac{1}{(3\theta)^n} I_{[-\theta<x_{(1)},\, x_{(n)}<2\theta]}$$

$\therefore \max\{-X_{(1)}, X_{(n)}/2\}$ is a Sufficient Statistic

and $\max\{-X_{(1)}, X_{(n)}/2\}$ is also a MLE for θ

$\therefore \max\{-X_{(1)}, X_{(n)}/2\}$ is the minimal Sufficient Statistic.

(b) $\max\{-X_{(1)}, X_{(n)}/2\}$'s distribution.

$$P\left(\max\{-X_{(1)}, X_{(n)}/2\} \le y\right) = P(-y \le X_i \le 2y,\ i=1,\cdots,n) = (y/\theta)^n$$

Let $Y = \max\{-X_{(1)}, X_{(n)}/2\}$, pdf of $Y = n\cdot\theta^{-n} y^{n-1}$

$$E[g(Y)] = \int_0^{\theta} g(y)\cdot n\theta^{-n} y^{n-1} dy \equiv 0 \Leftrightarrow g(\theta)\cdot\theta^{-n}\theta^{n-1} \equiv 0$$

$$\Leftrightarrow g(\theta) \equiv 0 \quad \therefore Y \text{ is a CSS for } \theta$$

$$E[Y] = \int_0^{\theta} y\cdot n\theta^{-n}\cdot y^{n-1} dy = \left[\theta^{-n}\frac{n}{n+1} y^{n+1}\right]_0^{\theta} = \frac{n}{n+1}\theta$$

$$\therefore \hat{\theta}^{UMVUE} = \frac{n+1}{n} Y \quad \checkmark\checkmark$$

8.14. 확률밀도함수가 $f(x;\theta) = \frac{1}{\theta}$, $x = 1,\cdots,\theta$ 로 주어지는 이산균등분포 $U\{1,\cdots,\theta\}$, $\theta\in\Omega$, $\Omega = \{1,2,\cdots\}$ 에서의 랜덤표본을 $X_1,\cdots,X_n$ 이라고 할 때 다음에 답하여라.

(a) $Y = \max_{1\le i\le n} X_i$ 가 $\theta\in\Omega$ 에 관한 완비충분통계량임을 밝혀라.

(b) θ의 전역최소분산불편추정량을 구하여라.

$$\prod_{i=1}^{n} f(x_i;\theta) = \theta^{-n} I_{[1\le x_{(1)},\, x_{(n)}\le\theta]}$$

$\therefore$ By the factorization theorem, $X_{(n)}$ is a SS for θ

$$P(X_{(n)} = y) = P(X_{(n)} \le y) - P(X_{(n)} \le y-1)$$

$$= \left(\frac{y}{\theta}\right)^n - \left(\frac{y-1}{\theta}\right)^n$$

$$E[g(Y)] = \sum_{y=0}^{\theta} g(y)\cdot\left\{\left(\frac{y}{\theta}\right)^n - \left(\frac{y-1}{\theta}\right)^n\right\} \equiv 0 \Leftrightarrow g(Y) \equiv 0$$

[Textbook] $\hookrightarrow g(\theta)\left\{\theta^n - (\theta-1)^n\right\} \overset{\theta}{\equiv} 0 \Rightarrow g(y) \overset{y}{\equiv} 0 \quad \therefore Y$ is a CSS for θ.

(b) $E[X_i] = \sum_{x=1}^{\theta} \frac{1}{\theta} \cdot x = \frac{1}{\theta} \cdot \frac{\theta(\theta+1)}{2} = \frac{\theta+1}{2}$

$\therefore \hat{\theta}^{UE} = 2X_i - 1$

$\hat{\theta}^{UMVUE} = E[\hat{\theta}^{UE} | X_{(n)}] = E[2X_1 - 1 | X_{(n)}] = 2[X_1 | X_{(n)}] - 1$

$E[\sum X_i] = E[nX_1] = n \cdot \frac{\theta+1}{2}$ $\therefore 2\bar{X} - 1$ is an unbiased estimator

$\hat{\theta}^{UMVUE} = E[2\bar{X} - 1 | X_{(n)}] = 2E[\bar{X} | X_{(n)}] - 1 = \frac{2}{n} E[\sum_{i=1}^{n} X_{(i)} | X_{(n)}] - 1$

$\frac{2}{n}\left\{ E[X_{(1)} + \cdots + X_{(n-1)} | X_{(n)}] + X_{(n)} \right\} - 1$

$X_{(1)} + \cdots + X_{(n-1)} = Z_1 + \cdots + Z_{n-1}$, $Z_j \sim U[1, X_{(n)}]$ ↓ X

$\therefore E[Z_1 + \cdots + Z_{n-1}] = \sum_{z=1}^{X_{(n)}} \frac{1}{X_{(n)}} \cdot z = \frac{X_{(n)}+1}{2}$

$$P_\theta(X_i = x | Y = y) = \frac{P_\theta(X = x, Y = y)}{P_\theta(X = x)} = \begin{cases} \dfrac{(1/\theta)[(y/\theta)^{n-1} - ((y-1)/\theta)^{n-1}]}{(y/\theta)^n - ((y-1)/\theta)^n}, & x = 1, \cdots, y-1 \\ \dfrac{(1/\theta)(y/\theta)^{n-1}}{(y/\theta)^n - ((y-1)/\theta)^n}, & x = y \end{cases}$$

$$\therefore P(X_i = x | Y = y) = \begin{cases} \dfrac{y^{n-1} - (y-1)^{n-1}}{y^n - (y-1)^n}, & x = 1, \cdots, y-1 \\ \dfrac{y^{n-1}}{y^n - (y-1)^n}, & x = y \end{cases}$$

$$E[X_i | Y = y] = \sum_{x=1}^{y} x P(X_i = x | Y = y) = \frac{y(y-1)}{2} \frac{y^{n-1} - (y-1)^{n-1}}{y^n - (y-1)^n} + y \frac{y^{n-1}}{y^n - (y-1)^n}$$

$$\therefore 2E[X_i | Y = y] - 1 = \frac{y^{n+1} - (y-1)^{n+1}}{y^n - (y-1)^n}$$

$$\therefore \hat{\theta}^{UMVUE} = \frac{Y^{n+1} - (Y-1)^{n+1}}{Y^n - (Y-1)^n}$$

If distribution is discrete, be careful!

8.15. 정규분포 $N(\mu, \sigma^2)$, $-\infty < \mu < +\infty$, $\sigma > 0$ 에서의 랜덤표본을 $X_1, \cdots, X_n$ ($n \geq 2$) 이라고 할 때 $\eta = \mu/\sigma$ 의 전역최소분산 불편추정량을 구하여라.

$f(x;\theta) = (2\pi\sigma^2)^{-1/2} \exp\left\{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right\}$, $\prod_{i=1}^{n} f(x_i;\theta) = (2\pi\sigma^2)^{-n/2} \exp\left\{-\frac{1}{2}\sum\left(\frac{x_i-\mu}{\sigma}\right)^2\right\}$

ΣX_i^2 and ΣX_i are SS for μ, σ by the Factorization Theorem.

⇔ $\Sigma(X_i - \bar{X})^2$ and $\bar{X}$ are SS for $\theta = (\mu, \sigma)$

$\bar{X} \sim N(\mu, \sigma^2/n)$, $\Sigma(X_i - \bar{X})^2/\sigma^2 \sim \chi^2(n-1) \overset{d}{=} \text{Gamma}\left(\frac{n-1}{2}, 2\right)$

Let $Y = \bar{X}$ and $Z = \Sigma(X_i - \bar{X})^2$

$$E[g(\theta)] = \int_{-\infty}^{+\infty} \int_{0}^{+\infty} g(y,z) \cdot pdf_Y(y) \cdot pdf_Z(z)\, dz\, dy$$

$$= \int_{-\infty}^{+\infty} \int_{0}^{+\infty} g(y,z) \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{1}{2}\left(\frac{z-\mu}{\sigma}\right)^2\right\} \cdot \frac{\Gamma\left(\frac{n-1}{2} \cdot 2\right)}{\Gamma\left(\frac{n-1}{2}\right)\Gamma(2)} \cdot y^{\frac{n-1}{2}-1} \exp\{-y/2\}\, dz\, dy \equiv 0$$

⇔ $g(Y,Z) \equiv 0$ (라플라스 변환의 유일성)

$\therefore \Sigma(X_i - \bar{X})^2$ and $\bar{X}$ are CSS for $\theta = (\mu, \sigma)^t$

or you can use the exponential family property and argue that ΣX_i^2 and ΣX_i are CSS.

$E[\bar{X}] = \mu$, $E\left[\frac{1}{n-1}\Sigma(X_i - \bar{X})^2\right] = \sigma^2$

$Y/\sigma^2 \sim \chi^2(n-1)$ (Here, $Y = \Sigma(X_i - \bar{X})^2$)

$\overset{d}{=} \text{Gamma}(a, \beta)$ $a = \frac{n-1}{2}$, $\beta = 2$

$$E\left[\frac{\sigma^2}{Y}\right] = \int_0^{\infty} y^{-1} \cdot \frac{1}{\Gamma(a)\beta^a} y^{a-1} \cdot \exp\{-y/\beta\}\, dy = (a-1)\cdot\beta$$

$$E[\sigma/\sqrt{Y}] = \int_0^{+\infty} y^{-1/2} \frac{1}{\Gamma(a)\beta^a} y^{a-1} \cdot \exp\{-y/\beta\}\, dy$$

$$= \frac{\Gamma(a-1/2)\beta^{a-1/2}}{\Gamma(a)\beta^a} \int_0^{\infty} \frac{1}{\Gamma(a-1/2)\beta^{a-1/2}} y^{a-1/2-1} \exp\{-y/\beta\}\, dy$$

$\therefore E\left[\frac{\Gamma(a-1/2)\beta^{a-1/2}}{\Gamma(a)\beta^a} \cdot Y^{-1/2}\right] = \sigma^{-1}$

$$\therefore \left(\frac{\mu}{\sigma}\right)^{UMVUE} = \left[\bar{X} \cdot \frac{\Gamma((n-2)/2)}{\Gamma((n-1)/2)\sqrt{2}} \left[\Sigma(X_i - \bar{X})^2\right]^{-1}\right]$$ ✓✓

8.16. 포아송분포 Poisson(θ), $\theta \geq 0$ 에서의 랜덤표본을 $X_1, \cdots, X_n$ 이라고 할 때

$\eta = P_\theta(X_1 \geq r)$ (r은 주어진 자연수)

의 전역최소분산불편추정량을 구하여라.

1. Find the CSS for θ ← exponential family / Factorization theorem + Show completeness.
2. Find the UE for θ — $\int_{x \in S_X} \hat{\theta}^{UE} \cdot pdf(\hat{\theta}^{UE} | \hat{\theta}^{CSS})\, dx$
3. Find the conditional expectation $E[\hat{\theta}^{UE} | \hat{\theta}^{CSS}] = \hat{\theta}^{UMVUE}$

3-1. Find function $g(\cdot)$ s.t. $E[g(\theta^{CSS})] = \theta$

$$f(x;\theta) = \frac{e^{-\theta}\theta^x}{x!} = \exp\{-\theta + x\log\theta - \log x!\}$$

$\therefore \Sigma X_i$ is a CSS for θ

$\Sigma \log x!$ is an ancillary statistic for θ

$\hat{\eta}^{UE} = I_{[r,\infty)}(x)$, $\hat{\eta}^{UMVUE} = E[I_{[r,\infty)}(x) | \Sigma X_i] = P(X_1 \geq r | \Sigma X_i)$

$$\hat{\eta}^{UMVUE} = \frac{P(X_1 \geq r, \Sigma X_i = t)}{P(\Sigma X_i = t)} \quad (t \geq r)$$

$$= \sum_{k=r}^{\infty} \frac{P(X_1 = k, \sum_2^n X_i = t-k)}{P(\Sigma X_i = t)} = \sum_{k=r}^{\infty} \frac{\frac{e^{-\theta}\theta^k}{k!} \cdot \frac{e^{-(n-1)\theta}\{(n-1)\theta\}^{t-k}}{(t-k)!}}{\frac{e^{-n\theta}(n\theta)^t}{t!}}$$

$$= \sum_{k=r}^{\infty} \frac{(n-1)^{t-k}}{n^t} \cdot \binom{t}{k} \quad (t \geq k)$$ ✓✓

★★★★★★

8.17. 정규분포 $N(\mu,\sigma^2)$, $-\infty<\mu<+\infty$, $\sigma>0$ 에서의 랜덤표본을 $X_1,\cdots,X_n$ $(n\geq 2)$ 이라고 할 때

$\eta = P_\theta(X_1>a)$ $(\theta=(\mu,\sigma)^t)$

의 전역최소분산불편추정량을 구하여라.

$\hat{\eta}^{UE} = I_{[a,\infty)}(x)$, CSS for θ : $(\Sigma X_i, \Sigma(X_i-\bar{X})^2)^t = (\bar{X}, S^2)$

* $\bar{X}=\frac{1}{n}\sum_{i=1}^n X_i$, $S^2=\frac{1}{n-1}\sum_{i=1}^n (X_i-\bar{X})^2$

ChatGPT solution : $\frac{X_1-\bar{X}}{S}\cdot\sqrt{\frac{n}{n-1}} \sim t_{n-1}$ $\quad \frac{\bar{X}-\mu}{\sigma}\sim N(0,1)$ $\quad \frac{(n-1)S^2}{\sigma^2}\sim \chi^2(n-1)$

$Z\sim N(0,1)$ Then $X_i-\bar{X}=(\mu+\sigma Z)-(\mu+\sigma\bar{Z})=\sigma(Z-\bar{Z})$

$S^2=(n-1)^{-1}\sum_{i=1}^n (X_i-\bar{X})^2/\sigma^2=(n-1)^{-1}\sum_{i=1}^n \sigma^2(Z-\bar{Z})^2$

$\therefore \frac{X_1-\bar{X}}{S}=\frac{(Z-\bar{Z})}{\sqrt{(n-1)^{-1}\sum_{i=1}^n\sigma(Z-\bar{Z})^2}}=\sqrt{(n-1)}\cdot\frac{Z_i-\bar{Z}}{\sqrt{\Sigma(Z_i-\bar{Z})^2}}$

$Var(Z_i-\bar{Z})=Var(Z_i)+Var(\bar{Z})-2Cov(Z_i,\bar{Z})=1+\frac{1}{n}-\frac{2}{n}=1-\frac{1}{n}$

$\therefore (Z_i-\bar{Z})\sim N(0,\frac{n-1}{n}) \Rightarrow U=\sqrt{\frac{n}{n-1}}\cdot(Z_i-\bar{Z})\sim N(0,1)$

$V=\sum_{i=1}^n (Z_i-\bar{Z})^2\sim\chi^2(n-1)$

Also, $U\perp V$ (independence)

$\therefore \frac{X_1-\bar{X}}{S}\sim\sqrt{n-1}\cdot\frac{Z_i-\bar{Z}}{\sqrt{\Sigma(Z_i-\bar{Z})^2}}\overset{d}{\equiv}\sqrt{n-1}\cdot\frac{\sqrt{\frac{n-1}{n}}\cdot U}{\sqrt{V}}$

$\therefore \frac{X_1-\bar{X}}{S}\sqrt{\frac{n}{n-1}}\overset{d}{\equiv}\frac{U}{\sqrt{V/(n-1)}}\sim t(n-1)$

$\Rightarrow X_1=\bar{X}+\sqrt{\frac{n}{n-1}}T\cdot S$, $T\sim t(n-1)$

$P(X_1>a\mid\bar{X},S)=P\left(\bar{X}+\sqrt{\frac{n}{n-1}}TS>a\right)=P\left(T>\sqrt{\frac{n}{n-1}}S^{-1}(a-\bar{X})\right)$

Let $F_{t_{n-1}}$ denote the cdf of the $t(n-1)$ distribution.

$P(X_1>a\mid\bar{X},S)=1-F_{t_{n-1}}\left(\sqrt{\frac{n}{n-1}}\cdot\frac{a-\bar{X}}{S}\right)$

[Textbook approach : using ChatGPT]

Step 1 : $\hat{\eta}^{UMVUE}=E[X_1>a\mid\bar{X},S]$

Step 2 : Rewrite using centered variable

Let $Y_i=\frac{X_i-\bar{X}}{S}$. Then $X_1>a=\frac{X_1-\bar{X}}{S}>\frac{a-\bar{X}}{S}=t$ $\quad\therefore Y_i>t$

$\hat{\eta}^{UMVUE}=P(Y_1>t\mid\bar{X},S^2)$

Step 3 : Key geometric fact

- $\Sigma Y_i=0$
- $\Sigma Y_i^2=n-1$

Thus, $Y=(Y_1,\cdots,Y_n)$ lies on the surface of a sphere in an $(n-1)$ dimension. ($\|Y\|=n-1$, which means, Y lies on a sphere of radius $\sqrt{n-1}$)

Moreover, conditional on $(\bar{X},S^2)$, Y is uniformly distributed on that sphere. This is a geometric core of the Beta distribution.

If $(Z_1,\cdots,Z_n)\sim N(0,I_k)$, $\frac{(Z_1,\cdots,Z_k)}{\sqrt{Z_1^2+\cdots+Z_k^2}}$ is uniformly distributed on the unit sphere in $\mathbb{R}^k$

$\frac{Y}{\sqrt{n-1}}\overset{d}{\equiv}\frac{(Z_1,\cdots,Z_{n-1})}{\sqrt{Z_1^2+\cdots+Z_{n-1}^2}} \Leftrightarrow \frac{Y_1^2}{n-1}\overset{d}{\equiv}\frac{Z_1^2}{Z_1^2+\cdots+Z_{n-1}^2}$

Now, $\wedge$ $Z_1^2\sim\chi_1^2$, $Z_1^2+\cdots+Z_{n-1}^2\sim\chi_{n-2}^2$

Hence $\frac{Y_1^2}{n-1}\sim Beta(\frac{1}{2},\frac{n-2}{2})$

Because the sphere is symmetric,

- Y_1 is equally likely to be positive or negative
- The sign is independent of Y_1^2

So we write : $\frac{Y_1}{\sqrt{n-1}}=sgn(Y_1)\sqrt{B}$, $B\sim Beta(\frac{1}{2},\frac{n-2}{2})$

$sgn(x)=\begin{cases}1 & x>0\\ 0 & x=0\\ -1 & x<0\end{cases}$

Then $P(Y_1>t)=\frac{1}{2}[1-sgn(t)P(B<\frac{t^2}{n-1})]$, $t=\frac{a-\bar{X}}{S}$

$=\frac{1}{2}-\frac{1}{2}sgn(t)P\left(B<\frac{(a-\bar{X})^2}{n-1\,S^2}\right)$ $\quad S^{2*}=\frac{1}{n}\Sigma(X_i-\bar{X})^2$, $S^2=\frac{n}{n-1}S^{2*}$

$=\frac{1}{2}-\frac{1}{2}sgn(t)B\left(\frac{(a-\bar{X})^2}{(n-1)S^2}\right)$

$=\frac{1}{2}-\frac{1}{2}sgn\left(\frac{a-\bar{X}}{S}\right)B\left(\frac{(a-\bar{X})^2}{(n-1)\cdot\frac{n}{n-1}S^{2*}}\right)$

$=\frac{1}{2}-\frac{1}{2}sgn(a-\bar{X})B\left(\frac{n(a-\bar{X})^2}{(n-1)^2S^{2*}}\right)$ ⇐ textbook

↳ ChatGPT 돌리니도 이와 같이만 할 것.

$X_1-\bar{X}=a^tX$, $a=e_1-n^{-1}\mathbb{1}^t$, $a^t\mathbb{1}=1-n^{-1}\mathbb{1}^t\mathbb{1}=1-n^{-1}n=0$

$(X_1-\bar{X})^2=(a^tX)^t(a^tX)=X^taa^tX$

$a^ta=(e_1-n^{-1}\mathbb{1})^t(e_1-n^{-1}\mathbb{1})=e_1^te_1-n^{-1}\mathbb{1}^te_1\cdot 2+n^{-2}\mathbb{1}^t\mathbb{1}$

$=1-n^{-1}\cdot 2+n^{-2}\cdot n$

$=1-n^{-1}=\frac{n-1}{n}$: constant

$x^taa^tX=(a^ta)\,x^ta(a^ta)^{-1}a^tX$

$=a^ta\,x^t\Pi_aX$, $\Pi_a=a(a^ta)^{-1}a^t$: projection to column space of 'a' (C_a)

$(n-1)s^2=X^t(I-\Pi_{\mathbb{1}})X=X^t(I-\Pi_{\mathbb{1}}-\Pi_a+\Pi_a)X$

$=X^t(I-\Pi_{\mathbb{1}}-\Pi_a)X+X^t\Pi_aX$

$rank(I-\Pi_{\mathbb{1}}-\Pi_a)=rank(I)-rank(\Pi_{\mathbb{1}})-rank(\Pi_a)$

$=n-2$

$rank(\Pi_a)=1$

$\therefore (n-1)s^2=X^t(I-\Pi_{\mathbb{1}}-\Pi_a)X+X^t\Pi_aX\overset{d}{\equiv}V_1\oplus V_2$, $V_1\sim\chi^2(n-2)$, $V_2\sim\chi^2(1)$

☆☆☆☆☆

$$\therefore W^2 = \left(\frac{X_1-\bar{X}}{S}\right)^2 \overset{d}{=} \frac{(a^t a)\, x^t \Pi_a x}{x^t(I-\Pi_1-\Pi_a+\Pi_a)x} = \frac{\frac{n-1}{n}V_1}{(V_1 \oplus V_2)/(n-1)}$$

$$= \frac{(n-1)^2 \cdot V_1}{n\cdot(V_1 \oplus V_2)} \sim \frac{(n-1)^2}{n}\, Beta\left(\frac{1}{2}, \frac{n-2}{2}\right)$$

$$\therefore P(W^2 > w^2) = \frac{1}{2} - \frac{1}{2}\, Sgn(w)\, P(W^2 \le w^2)$$

$$= \frac{1}{2} - \frac{1}{2}\, Sgn(w)\, P\left(\frac{n}{(n-1)^2}\cdot W^2 \le \frac{n}{(n-1)^2} w^2\right)$$

$$= \frac{1}{2} - \frac{1}{2}\, Sgn(w)\, Beta\left(\frac{n}{(n-1)^2} w^2\right)$$

$$= \frac{1}{2} - \frac{1}{2}\, Sig\left(\frac{a-\bar{X}}{S}\right) Beta\left(\frac{n}{(n-1)^2}\cdot\left(\frac{a-\bar{X}}{S}\right)^2\right)$$

$$= \frac{1}{2} - \frac{1}{2}\, Sig\left(\frac{a-\bar{X}}{S}\right) Beta\left(\frac{n(a-\bar{X})^2}{(n-1)^2 S^2}\right)$$

8.18. 두 개의 모수를 갖는 지수분포 $Exp(\mu,\sigma)$, $-\infty<\mu<+\infty$, $\sigma>0$ 에서의 랜덤표본을 $X_1,\cdots,X_n$ $(n\ge 2)$ 이라고 할 때

$$\eta = P_\theta(X_1 > a) \quad (\theta = (\mu,\sigma)^t)$$

의 전역최소분산 불편추정량을 구하여라.

$$f(x;\theta) = \frac{1}{\sigma}\exp\left\{-\frac{x-\mu}{\sigma}\right\} I_{[\mu,\infty)}(x) = \exp\left\{-\log\sigma - \frac{1}{\sigma}x + \frac{\mu}{\sigma}\right\} I_{[\mu,\infty)}(x)$$

$\therefore$ by the factorization theorem ΣX_i and $X_{(1)}$ are sufficient statistic

How can I show completeness of ΣX_i and $X_{(1)}$

$$\hat{\eta}^{UMVUE} = E[\hat{\eta}^{UE} \mid \Sigma X_i, X_{(1)}] = E[I_{(a,\infty)}(X) \mid \Sigma X_i, X_{(1)}] = P(X_1 > a \mid \Sigma X_i, X_{(1)})$$

$$= \frac{P(X_1 > a, \Sigma X_i = y_1, X_{(1)} = y_2)}{P(\Sigma X_i = y_1, X_{(1)} = y_2)}$$

what is an ancillary statistic for θ? $(X_i-\mu)/\sigma \sim Exp(1)$

$$\frac{X_1-\mu}{\sigma} \sim Gamma(1,1) \quad \frac{\Sigma(X_i-\mu)}{\sigma} \sim Gamma(n,1) \quad n\cdot X_{(1)} \sim Exp(1)$$

$$\sum_{i=1}^n X_i = \sum_{i=1}^n X_{(i)} = nX_{(1)} + (n-1)(X_{(2)}-X_{(1)}) + \cdots + 1\,(X_{(n)}-X_{(n-1)})$$
$$= Exp(1) \oplus \cdots \oplus Exp(1)$$

$$P(X_1 > a \mid \Sigma X_i = y_1, X_{(1)} = y_2) \Rightarrow P(X_1 - X_{(1)} > a \mid \Sigma X = y_1, X_{(1)} = y_2)$$

연습문제 (8.11)로부터 $(S,T)^t = (X_{(1)}, \sum_{i=1}^n (X_i - X_{(1)}))^t$ 는 $(\mu,\sigma)\in\mathbb{R}\times\mathbb{R}_+$ 에 관한 CSS이다. S와 T는 서로 독립이며 각각의 분포는 다음과 같다.

$$n(S-\mu)/\sigma \sim Exp(1), \quad \sum_{i=1}^n (X_i - X_{(1)})^t = \sum_{i=1}^n (X_{(i)} - X_{(1)})^t$$

$$= \sum_{i=2}^n (n-i+1)(X_{(i)} - X_{(1)})^t \sim Exp(1) \oplus \cdots \oplus Exp(1)$$

$$= Gamma(n-1,1)$$

$$* (n-r-1)(X_{(r)} - X_{(r-1)}) \sim Exp(1)$$

(i) $(X_i-\mu)/\sigma \overset{iid}{\sim} Exp(1)$ 이므로

$\frac{X_1-S}{T} = \frac{X_1 - X_{(1)}}{X_1+\cdots+X_n - nX_{(1)}}$ 는 $\theta\in(\mu,\sigma)^t\in(-\infty,+\infty)^t\times(0,+\infty)$ 에 관한 보조통계량이다. 따라서 정리 3.2.2 으로부터 이 통계량은 완비충분통계량과 독립이다.

$$P(X_1 > a \mid Y = y) = P\left(\frac{X_1-S}{T} > \frac{a-y_1}{y_2} \mid Y = y\right) = P\left(\frac{X_1-S}{T} > \frac{a-y_1}{y_2}\right)$$

여기서 $X_1 - S = X_1 - X_{(1)} \ge 0$ 이므로

$$P\left(\frac{X_1-S}{T} > w\right) = \begin{cases} 1 & w < 0 \\ P\left(\frac{X_1-S}{T} > w_+\right), & w \ge 0 \end{cases} \qquad w_+ = \max(w,0)$$

(ii) $X_{(1)} = \min(X_1, \min_{2\le i\le n} X_i)$ 이므로,

$Z_1 = X_2, \cdots, Z_{n-1} = X_n$, $m = n-1$, $a \wedge b = \min(a,b)$ 라고 하면

$$\frac{X_1 - X_{(1)}}{X_1 + \cdots + X_m - nX_{(1)}} = \frac{X_1 - X_1 \wedge Z}{X_1 + Z_1 + \cdots + Z_m - (m+1)X_1 \wedge Z_{(1)}}$$

따라서

$$P\left(\frac{X_1 - X_{(1)}}{X_1 + \cdots + X_n - nX_{(1)}} > w_+\right)$$

$$= P\left(\frac{X_1 - X_1 \wedge Z_1}{X_1 + Z_1 + \cdots + Z_m - (m+1)X_1 \wedge Z_{(1)}} > w_+\right)$$

$$= P\left(\frac{X_1 - Z_{(1)}}{X_1 + Z_1 + \cdots + Z_m - (m+1)X_1 \wedge Z_{(1)}} > w_+,\ X_1 > Z_{(1)}\right)$$

또한 (i)에서 밝혔듯이 이 확률은 분포형태량에 관한 확률이므로 μ, σ에 의존하지 않는다. 그러므로 이제 이 확률의 계산에서 $\mu=0, \sigma=1$ 이라고 가정하도록 한다. 이러한 경우에

$$X_1 \sim Exp(1),\ mZ_{(1)} \sim Exp(1),\ \sum_{i=1}^{m}(Z_i - Z_{(1)}) \sim Gamma(m-1, 1)$$

이고, $X_1, Z_{(1)}, \sum_{i=1}^{m}(Z_i - Z_{(1)})$ 는 서로 독립이다. 이로부터

$$P(X_1 - Z_{(1)} > x) = \frac{m}{m+1} e^{-x} \quad x \ge 0$$

$$= P(X_1 > x + Z) = 1 - P(X_1 < x + Z)$$

$$= 1 - \int_0^\infty \int_0^{x+z} e^{-x}\,dx \cdot me^{-mz}\,dz$$

$$= 1 - \int_0^\infty \left[-e^{-x}\right]_0^{x+z_{(1)}} \cdot me^{-mz}\,dz$$

$$= 1 - \int_0^\infty (1 + e^{-x-z})\, me^{-mz}\,dz$$

$$= 1 - \int_0^\infty me^{-mz} - me^{-x} \cdot e^{-(m+1)z}\,dz$$

$$= 1 - \left[-e^{-mz} + \frac{m}{m+1} e^{-x} \cdot e^{-(m+1)z}\right]_0^\infty$$

$$= 1 - \left[-0 + 0 - \left(-1 + \frac{m}{m+1} e^{-x}\right)\right]$$

$$= 1 - 1 + \frac{m}{m+1} e^{-x} = \frac{m}{m+1} e^{-x}$$

임을 알 수 있고, 위의 확률을 $V = \sum_{i=1}^{m}(Z_i - Z_{(1)})$ 를 이용하여 다음과 같이 나타낼 수 있다.

$$P\left(\frac{X_1 - Z_{(1)}}{X_1 + Z_1 + \cdots + Z_m - (m+1)Z_{(1)}} > w_+,\ X_1 > Z_1\right)$$

$$= P\left(X_1 - Z_{(1)} > w_+\left(X_1 + Z_1 + \cdots + Z_m - (m+1)Z_{(1)}\right),\ X_1 > Z_1\right)$$

$$= P\left(X_1 - Z_{(1)} - w_+(X_1 - Z_{(1)}) > w_+(Z_1 + \cdots + Z_m - mZ_{(1)}),\ X_1 > Z_1\right)$$

$$= P\left((X_1 - Z_{(1)})(1 - w_+) > w_+ \cdot V,\ X_1 > Z_1\right)$$

$$= P\left(X_1 - Z_{(1)} > \left(\frac{w_+}{1-w_+} \cdot V\right),\ X_1 > Z_1\right)$$

$$= \begin{cases} E\left[\frac{m}{m+1} \exp\left\{-\frac{w_+}{1-w_+} V\right\}\right], & 0 \le w_+ \le 1 \\ 0, & w_+ \ge 1 \end{cases}$$

여기에서 $V = \sum_{i=1}^{n}(Z_i - Z_{(1)}) \sim Gamma(m-1, 1)$

임을 이용하면 다음이 성립함을 알 수 있다.

$$E\left[\frac{m}{m+1} \exp\left\{-\frac{w_+}{1-w_+} V\right\}\right]$$

$$= \int_0^\infty \frac{m}{m+1} \cdot \exp\left\{-\frac{w_+}{1-w_+} \cdot v\right\} \cdot \frac{1}{\Gamma(m-1)\cdot 1} v^{m-2} \exp\left\{-\frac{v}{1}\right\} dv$$

$$= \int_0^\infty \frac{m}{m+1} \cdot \frac{1}{\Gamma(m-1)} v^{m-2} \cdot \exp\left\{-\frac{w_+ + 1 - w_+}{1-w_+} v\right\} dv$$

$$= \frac{m}{m+1}(1-w_+)^{m-1} \int_0^\infty \frac{1}{\Gamma(m-1)(1-w_+)^{m-1}} v^{m-2} \exp\left\{-\frac{v}{1-w_+}\right\} dv$$

$$= \frac{m}{m+1}(1-w_+)^{m-1}$$

$$\therefore \hat{w}_{UMVUE} = \begin{cases} 1, & a - s < 0 \\ \frac{m}{m+1}(1-w_+)^{m-1} = \frac{n-1}{n}\left(1 - \frac{a-s}{T}\right)^{n-2}, & 0 \le a - s \le T \\ 0, & a - s \ge T \end{cases}$$

□

문제 1.20, 예 5.3.2 응용

★ 8.19. 포아송분포 Poisson(θ), $\theta \geq 0$ 에서의 랜덤표본을 $X_1, \cdots, X_n$ 이라고 할 때

$$\hat{\theta}_n^{(1)} = \sum_{i=1}^n X_i/n \,, \quad \hat{\theta}_n^{(2)} = \sum_{i=1}^n (X_i - \bar{X})^2/n$$

에 대하여 점근상대효율성 $ARE\left(\left\{\hat{\theta}_n^{(1)}\right\}, \left\{\hat{\theta}_n^{(2)}\right\}\right)$을 구하여라.

$\hat{\theta}_n^{(1)} = \bar{X}$ $\therefore \sqrt{n}(\bar{X}-\theta) \longrightarrow N(0,\theta)$

$\hat{\theta}_n^{(2)} = \frac{1}{n}\sum (X_i-\bar{X})^2 = \hat{\sigma}^2$

$\sqrt{n}(\hat{\sigma}^2 - \sigma^2) \longrightarrow (0, (\rho_4+2)\sigma^4)$

$\rho_4 = C_Z^{(4)}(t=0) = \dfrac{C_X^{(4)}(t=0)}{\sigma^4}$

$f(x;\theta) = \dfrac{e^{-\theta}\theta^x}{x!}$, $E[e^{tX}] = \sum_{x=0}^{\infty} \dfrac{e^{-\theta}\cdot(e^t\theta)^x}{x!} = e^{-\theta}\cdot e^{e^t\theta} = \exp\{\theta(e^t-1)\}$

$\therefore M_X(t) = \exp\{\theta(e^t-1)\}$, $C_X(t) = \theta(e^t-1)$ $\therefore C_X^{(4)}(t) = \theta e^t$, $C_X^{(4)}(t=0) = \theta$

$\therefore \sqrt{n}(\hat{\sigma}^2 - \sigma^2) \longrightarrow N(0, (\theta^4+2)\cdot\theta^2) \overset{d}{\equiv} N(0, \theta+2\theta^2)$

$\therefore ARE\left(\left\{\hat{\theta}_n^{(1)}, \hat{\theta}_n^{(2)}\right\}\right) = avar(\hat{\theta}^{(1)})^{-1} / avar(\hat{\theta}^{(2)})^{-1}$ * avar: asymptotic variance.

$= (\theta+2\theta^2)/\theta$

$= 1+2\theta \geq 1$ ✓✓

$\therefore \hat{\theta}_n^{(1)}$ is asymptotically a better estimator than $\hat{\theta}_n^{(2)}$

8.20. 정규분포 $N(\mu,\sigma^2)$, $-\infty < \mu < +\infty$, $\sigma > 0$ 에서의 랜덤표본을 $X_1, \cdots, X_n$ $(n \geq 2)$ 이라고 할 때 다음에 답하여라.

(a) σ의 전역최소분산 불편추정량 $\hat{\sigma}_n^{UMVUE}$을 구하여라.

(b) 표본표준편차 $S_n = \sqrt{\sum_{i=1}^n (X_i-\bar{X})^2/(n-1)}$ 의 전역최소불편추정량 $\hat{\sigma}_n^{UMVUE}$에 대한 점근상대효율성 $ARE\left(\{S_n\}, \{\hat{\sigma}_n^{UMVUE}\}\right)$을 구하여라.

(c) σ의 불편추정량 $\hat{\sigma}_n^{UE}$의 분산에 대한 정보량 부등식을 구하여라.

(참고: 점근상대효율성의 계산에서 다음 근사식을 사용한다. 이러한 근사식은 부록 I의 〈예 I.3.1〉과 〈정리 I.7.1〉로부터 밝힐 수 있다.)

$$\Gamma(m+1) = m^{m+1/2} e^{-m} \sqrt{2\pi}(1+r_m), \quad \sqrt{m}\, r_m \longrightarrow 0$$

$$(1+a/n)^n = e^a(1+R_n), \quad \sqrt{n}\, R_n \longrightarrow 0$$

(a) $\hat{\sigma}^{2\,UE} = \dfrac{1}{n-1}\sum (X_i-\bar{X})^2$, $\sum (X_i-\bar{X})^2$ is a CSS for $\theta = (\mu,\sigma)^t$

$S_n^2 = \dfrac{1}{n-1}\sum (X_i-\bar{X})^2$, $(n-1)S^2/\sigma^2 \sim \chi^2(n-1) \overset{d}{\equiv} Gamma\left(\frac{n-1}{2}, 2\right)$

$$E\left[\left\{\frac{(n-1)S^2}{\sigma^2}\right\}^{1/2}\right] = \int_0^\infty v^{1/2}\frac{1}{\Gamma(a)\beta^a} v^{a-1}\exp\{-v/\beta\}dv$$

$$= \frac{\Gamma(a+1/2)}{\Gamma(a)}\beta^{1/2}\int_0^\infty \frac{1}{\Gamma(a+\frac{1}{2})\cdot\beta^{a+1/2}} v^{a+1/2-1}\exp\{-v/\beta\}dv$$

$$= \frac{\Gamma(n/2)}{\Gamma((n-1)/2)}2^{1/2} = c_n$$

$\therefore E\left[c_n^{-1}\cdot\left\{(n-1)S^2\right\}^{1/2}\right] = \sigma$

$\hat{\sigma}^{UE} = c_n^{-1}\cdot\left\{(n-1)S^2\right\}^{1/2}$, $c_n = \dfrac{\Gamma(n/2)}{\Gamma((n-1)/2)}\cdot\sqrt{2}$

$\hat{\sigma}^{UMVUE} = E[\hat{\sigma}^{UE}\,|\,\bar{X}, S^2] = c_n^{-1}\sqrt{n-1}\,S = \dfrac{\Gamma((n-1)/2)}{\Gamma(n/2)}\cdot\sqrt{\dfrac{n-1}{2}}\cdot S$ ✓✓

* $S^2 = \frac{1}{n-1}\sum_{i=1}^n (X_i-\bar{X})^2$

(b) $\sqrt{n}(S_n-\sigma) \longrightarrow N(0, (\rho_4+2)\cdot\sigma^2/4)$

$\rho_4 = C_X^{(4)}(t=0) = C_Z^{(4)}(t=0)/\sigma^4$

$M_X(t) = \exp(\mu t + \frac{1}{2}t^2\sigma^2)$

$C_X(t) = \mu t + \frac{1}{2}t^2\sigma^2$

$\therefore C_X^{(4)}(t) = 0$

$\therefore \sqrt{n}(S_n-\sigma) \longrightarrow N(0, \frac{1}{2}\sigma^2)$

$\sqrt{n}(\hat{\sigma}^{UMVUE} - \sigma) = \sqrt{n}(\hat{\sigma}^{UMVUE} - S_n) + \sqrt{n}(S_n-\sigma)$

$\sqrt{n}(\hat{\sigma}^{UMVUE} - S_n) = \sqrt{n}\left(\left\{\dfrac{\Gamma((n-1)/2)}{\Gamma(n/2)}\cdot\sqrt{\dfrac{n-1}{2}} - 1\right\}S_n\right)$

$$\frac{\Gamma((n-1)/2)}{\Gamma(n/2)} = \frac{\Gamma\left(\frac{n-2}{2}+\frac{1}{2}\right)}{\Gamma\left(\frac{n-1}{2}+\frac{1}{2}\right)} = \frac{\Gamma(t+1/2)}{\Gamma(s+1/2)} = \frac{t^{t+1/2}e^{-t}\sqrt{2\pi}(1+r_t)}{s^{s+1/2}e^{-s}\sqrt{2\pi}(1+r_s)}$$

$$= \frac{t^{t+1/2}}{s^{t+1}}\cdot e^{-t+s}\frac{(1+r_t)}{(1+r_s)} = \left(\frac{t}{s}\right)^t s^{-1/2}e^{-t+s}\frac{\sqrt{t}(1+r_t)}{\sqrt{s}(1+r_s)}$$

$$= \left(\frac{t}{t+1/2}\right)^t\left(\frac{n-1}{2}\right)^{-1}e^{1/2}\frac{\sqrt{t}(1+r_t)}{\sqrt{s}(1+r_s)} = \left(1+\frac{1/2}{t}\right)^{-t}\left(\frac{n-1}{2}\right)^{-1}e^{1/2}\frac{\sqrt{t}(1+r_t)}{\sqrt{s}(1+r_s)}$$

$\therefore \dfrac{\Gamma((n-1)/2)}{\Gamma(n/2)}\sqrt{\dfrac{n-1}{2}} = \left(1+\frac{1/2}{t}\right)^{-t}\left(\frac{n-1}{2}\right)^{-1}e^{1/2}\dfrac{\sqrt{t}(1+r_t)}{\sqrt{s}(1+r_s)} \overset{n\to\infty}{\longrightarrow} e^{-1/2}\cdot e^{1/2} = 1$

$\therefore \sqrt{n}(\hat{\sigma}^{UMVUE} - S_n) \to 0 \Rightarrow \sqrt{n}(\hat{\sigma}^{UMVUE} - \sigma) \underset{n\to\infty}{\overset{d}{\longrightarrow}} \sqrt{n}(S_n-\sigma) \underset{n\to\infty}{\longrightarrow} N(0, \frac{1}{2}\sigma^2)$

$\therefore ARE\left(\left\{S_n\right\}, \left\{\hat{\sigma}^{UMVUE}\right\}\right) = 1$ ✓✓

(c) RCLB of $\hat{\theta}^{UE} = \left[\frac{\partial}{\partial\theta}E_\theta[\hat{\theta}^{UE}]\right]^t\left\{n\,I(\theta)\right\}^{-1}\left[\frac{\partial}{\partial\theta}E_\theta[\hat{\theta}^{UE}]\right]^t$

$= \left[n\,I(\theta)\right]^{-1} = \left[n\cdot E\left[-\frac{\partial^2}{\partial\theta}\log f(x;\theta)\right]\right]^{-1}$

$\Leftrightarrow \left[n\cdot Var\left(\frac{\partial}{\partial\theta}\log f(x;\theta)\right)\right]$

$f(x;\theta) = \exp\left\{-\frac{1}{2}\log(2\pi\sigma^2) - \frac{1}{2\sigma^2}(x-\mu)^2\right\}$

$\log f(x;\theta) = -\frac{1}{2}\log(2\pi\sigma^2) - \frac{1}{2\sigma^2}(x-\mu)^2$

$\frac{\partial}{\partial\theta}\log f(x;\theta) = \left[+\frac{1}{\sigma^2}(x-\mu) \quad -\frac{1}{2\sigma^2}+\frac{1}{2\sigma^4}(x-\mu)^2\right]^t$

$$\frac{\partial^2}{\partial\theta}\log f(x;\theta) = \begin{bmatrix} -\sigma^{-2} & -\frac{1}{\sigma^4}(x-\mu) \\ -\frac{1}{\sigma^4}(x-\mu) & \frac{1}{2\sigma^4} - \frac{1}{\sigma^6}(x-\mu)^2 \end{bmatrix}$$

$$E\left[\frac{\partial^2}{\partial\theta}\log f(x;\theta)\right] = \begin{bmatrix} \sigma^{-2} & 0 \\ 0 & \frac{1}{2\sigma^4} - \frac{1}{\sigma^4} \end{bmatrix}$$

$$I(\theta) = \begin{bmatrix} \sigma^{-2} & 0 \\ 0 & \frac{1}{2}\sigma^{-4} \end{bmatrix}$$

$$I(\theta) = \begin{bmatrix} \sigma^{-2} & 0 \\ 0 & \frac{1}{2}\sigma^{-4} \end{bmatrix}$$

$$\therefore I(\theta)^{-1} = \frac{1}{\frac{1}{2}\sigma^{-6}} \begin{bmatrix} \frac{1}{2}\sigma^{-4} & 0 \\ 0 & \sigma^{-2} \end{bmatrix} = \begin{bmatrix} \sigma^2 & 0 \\ 0 & 2\sigma^4 \end{bmatrix}$$

$$\therefore (nI(\theta))^{-1} = \begin{bmatrix} \sigma^2/n & 0 \\ 0 & 2\sigma^4/n \end{bmatrix}$$

$$Var(\hat{\sigma}^{UE}) = \begin{bmatrix} 0 & 1 \end{bmatrix} \left[n \begin{pmatrix} 1/\sigma^2 & 0 \\ 0 & 2/\sigma^2 \end{pmatrix} \right]^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{n}\frac{\sigma^2}{2}$$

parameters are (μ, σ) instead of (μ, σ^2)

8.21. <정리 8.4.1>의 정보량 부등식에서 주어지는 분산의 하한은 모수의 일대일 변환에 무관한 것을 밝혀라. 즉 $\eta = g(\theta)$이고 g가 일대일이고 미분가능한 함수일 때

$$\left(\frac{\partial}{\partial\theta} E_\theta(\hat{\eta}_n)\right)^t \left[n\, Var_\theta\left(\frac{\partial}{\partial\theta} \log f(x_i;\theta)\right)\right]^{-1} \left(\frac{\partial}{\partial\theta} E_\theta(\hat{\eta}_n)\right)$$

$$= \left(\frac{\partial}{\partial\xi} E_\xi(\hat{\eta}_n)\right)^t \left[n\, Var_\xi\left(\frac{\partial}{\partial\xi} \log f(x_i; g^{-1}(\xi))\right)\right]^{-1} \left(\frac{\partial}{\partial\xi} E_\xi(\hat{\eta}_n)\right)$$

$$\frac{\partial}{\partial\theta} E_\theta(\hat{\eta}_n) = \frac{\partial}{\partial\theta} \int_{x\in S_x} \hat{\eta}_n f(x;\theta)\, dx = \frac{\partial}{\partial\xi}\cdot\frac{\partial\xi}{\partial\theta} \int \hat{\eta}_n f(x;\xi)\, dx = \frac{\partial}{\partial\xi}\cdot\frac{\partial\xi}{\partial\theta} \int \hat{\eta}_n \cdot f(x;\xi)\, dx$$

$$\Rightarrow \frac{\partial}{\partial\theta} E_\theta[\hat{\eta}_n] = \frac{\partial}{\partial\xi}\cdot\frac{\partial\xi}{\partial\theta} E_\xi[\hat{\eta}_n]$$ ✓✓

[Textbook Approach]

$$Var_\theta\left(\frac{\partial}{\partial\theta}\log f(x_i;\theta)\right) = Var\left(\frac{\partial\xi}{\partial\theta}\cdot\frac{\partial}{\partial\xi}\log f(x;\theta)\right) = \left(\frac{\partial\xi}{\partial\theta}\right) Var\left(\frac{\partial}{\partial\xi}\log f(x;\theta)\right)\left(\frac{\partial\xi}{\partial\theta}\right)^t$$

$$\left(\frac{\partial}{\partial\theta} E_\theta[\hat{\eta}_n]\right)^t \left(n\, Var_\theta\left(\frac{\partial}{\partial\theta}\log f(x;\theta)\right)\right)^{-1} \left(\frac{\partial}{\partial\theta} E_\theta(\hat{\eta}_n)\right)$$

$$= \left(\frac{\partial\xi}{\partial\theta}\frac{\partial}{\partial\xi} E_\xi[\hat{\eta}_n]\right)^t \left(n\,\frac{\partial\xi}{\partial\theta}\, Var\left(\frac{\partial}{\partial\xi}\log f(x;g^{-1}(\xi))\right)\left(\frac{\partial\xi}{\partial\theta}\right)^t\right)^{-1} \left(\frac{\partial\xi}{\partial\theta}\frac{\partial}{\partial\xi} E_\xi(\hat{\eta}_n)\right)$$

$$= \left(\frac{\partial}{\partial\xi} E_\xi[\hat{\eta}_n]\right)^t \left(\frac{\partial\xi}{\partial\theta}\right)^t \left(\frac{\partial\xi}{\partial\theta}\right)^{-1} \left[n\, Var\left(\frac{\partial}{\partial\xi}\log f(x;g^{-1}(\xi))\right)\right]^{-1} \left(\frac{\partial\xi}{\partial\theta}\right)^{-t} \left(\frac{\partial\xi}{\partial\theta}\right) \left(\frac{\partial}{\partial\xi} E_\xi(\hat{\eta}_n)\right)$$

shape 추리

$$= \left(\frac{\partial}{\partial\xi} E_\xi[\hat{\eta}_n]\right)^t \left[n\, Var\left(\frac{\partial}{\partial\xi}\log f(x;g^{-1}(\xi))\right)\right]^{-1} \left(\frac{\partial}{\partial\xi} E_\xi(\hat{\eta}_n)\right)$$

★ 8.22. 연습문제 (8.15)에서 $\eta = \eta(\theta)$의 불편추정량 $\hat{\eta}_n^{UE}$의 분산에 대한 정보량 부등식을 구하고 $\eta = \eta(\theta)$의 최대가능도추정량 $\hat{\eta}_n^{MLE}$의 극한분포를 구하여라.

1. Information inequality 2. limiting distribution of MLE

$$f(x;\theta) = \exp\left\{-\frac{1}{2}\log(2\pi\sigma^2) - \frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right\}$$

$$= \exp\left\{-\frac{1}{2}\log(2\pi) - \log(\sigma) - \frac{1}{2}\left(\frac{x^2 - 2\mu x + \mu^2}{\sigma^2}\right)\right\}$$

$$= \exp\left\{-\frac{1}{2}\log(2\pi) + \log(\eta_2) - \frac{1}{2}\left(x^2\eta_2^2 - 2\eta_1\eta_2 x + \eta_1^2\right)\right\}, \quad \eta_1 = \frac{\mu}{\sigma^2},\ \eta_2 = \frac{1}{\sigma}$$

$$\log f(x;\theta) = -\frac{1}{2}\log(2\pi) + \log(\eta_2) - \frac{1}{2}\left(x^2\eta_2^2 - 2\eta_1\eta_2 x + \eta_1^2\right)$$

$$\frac{\partial}{\partial\theta}\log f(x;\theta) = \begin{bmatrix} \eta_2 x - \eta_1 & \frac{1}{\eta_2} - x^2\eta_2 + \eta_1 x \end{bmatrix}$$

$$\frac{\partial^2}{\partial\theta^2}\log f(x;\theta) = \begin{bmatrix} -1 & x \\ x & -\eta_2^{-2} - x^2 \end{bmatrix}$$

$$E\left[-\frac{\partial^2}{\partial\theta^2}\log f(x;\theta)\right] = \begin{bmatrix} 1 & -\mu \\ -\mu & \sigma^2 + \sigma^2 + \mu^2 \end{bmatrix}$$

$$I(\theta) = \begin{bmatrix} 1 & -\mu \\ -\mu & 2\sigma^2 + \mu^2 \end{bmatrix}, \quad I(\theta)^{-1} = \frac{1}{2\sigma^2}\begin{bmatrix} 2\sigma^2 + \mu^2 & \mu \\ \mu & 1 \end{bmatrix}$$

$$= \frac{1}{2\sigma^2}\begin{bmatrix} 2\sigma^2 + \mu^2 & \mu \\ \mu & 1 \end{bmatrix}$$

$$(nI(\theta))^{-1} = \frac{1}{2\sigma^2}\begin{bmatrix} 2\sigma^2 + \mu^2 & \mu \\ \mu & 1 \end{bmatrix}$$

$$Var(\hat{\eta}^{UE}) = \begin{bmatrix} 1 & 0 \end{bmatrix}\left(\frac{\partial}{\partial\theta} E_\theta[\hat{\eta}_n^{UE}]\right)\left[nI(\theta)\right]^{-1}\left(\frac{\partial}{\partial\theta} E_\theta[\hat{\eta}_n^{UE}]\right)^t\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \frac{2\sigma^2 + \mu^2}{2\sigma^2 n} = \frac{1}{n}\left(1 + \frac{\mu^2}{2\sigma^2}\right)$$ ✓✓

[Textbook Approach]

$$Var_\theta(\hat{\eta}_n^{UE}) \geq \left(\frac{\partial}{\partial\theta}\eta(\theta)\right)^t \left[n\, Var_\theta\left(\frac{\partial}{\partial\theta}\log f(x;\theta)\right)\right]^{-1} \left(\frac{\partial}{\partial\theta}\eta(\theta)\right)$$

$$= \begin{bmatrix} 1/\sigma \\ -\mu/\sigma^2 \end{bmatrix}^t \left[n \begin{pmatrix} 1/\sigma^2 & 0 \\ 0 & 2/\sigma^2 \end{pmatrix}\right]^{-1} \begin{bmatrix} 1/\sigma \\ -\mu/\sigma^2 \end{bmatrix}$$

$$= \frac{1}{n}\left(1 + \frac{1}{2}\frac{\mu^2}{\sigma^2}\right)$$

$$\hat{\mu}^{MLE} = \bar{x} = \frac{1}{n}\sum_{i=1}^n x_i, \quad \hat{\sigma}^{2\,MLE} = \frac{1}{n}\sum_{i=1}^n (x_i - \bar{x})^2$$

$$\therefore \left(\frac{\mu}{\sigma}\right)^{MLE} = \frac{\bar{x}}{\left(\frac{1}{n}\sum(x_i - \bar{x})^2\right)}$$

* parameter: (μ, σ)

$$\sqrt{n}\left(\hat{\theta}^{MLE} - \theta\right) \longrightarrow N(0, I(\theta)^{-1}), \quad I(\theta) = \begin{bmatrix} \sigma^2 & 0 \\ 0 & \frac{1}{2}\sigma^2 \end{bmatrix}$$

$$\sqrt{n}\left(g(\hat{\theta}^{MLE}) - g(\theta)\right) \longrightarrow N(0, \dot{g}(\theta)^t I(\theta)^{-1}\dot{g}(\theta))$$

$$\theta = \begin{pmatrix} \mu & \sigma \end{pmatrix} \quad g(\theta) = \frac{\mu}{\sigma} \quad \dot{g}(\theta) = \begin{bmatrix} \frac{1}{\sigma} & -\frac{\mu}{\sigma^2} \end{bmatrix}^t$$

$$avar = \dot{g}(\theta)^t I(\theta)^{-1}\dot{g}(\theta) = \frac{1}{n}\left(1 + \frac{\mu^2}{2\sigma^2}\right)$$ ✓✓

8.23. 연습문제 (8.17)에서 $a=0$ 인 경우에 $\eta=\eta(\theta)$의 불편추정량 $\hat{\eta}_n^{UE}$의 분산에 대한 정보량 부등식을 구하고, $\eta=\eta(\theta)$의 최대가능도추정량 $\hat{\eta}_n^{MLE}$의 극한분포를 구하여라.

$X\sim N(\mu,\sigma^2)$, $\eta=P_\theta(X_1>a)=P_\theta(X_1>0)$, $\hat{\eta}^{UE}=I_{(0,\infty)}(X)$

Information Inequality : $\left(\frac{\partial}{\partial\theta}E_\theta[\hat{\eta}_n^{UE}]\right)^t\left[n\cdot Var\left(\frac{\partial}{\partial\theta}\log f(x;\theta)\right)\right]^{-1}\left(\frac{\partial}{\partial\theta}E_\theta[\hat{\eta}_n^{UE}]\right)$

$=\left(\frac{\partial}{\partial\theta}P_\theta(X_1>0)\right)^t\left[n\cdot Var\left(\frac{\partial}{\partial\theta}\log f(x;\theta)\right)\right]^{-1}\left(\frac{\partial}{\partial\theta}P_\theta(X_1>0)\right)$

$P_\theta(X_1>0)=P\left(\frac{X-\mu}{\sigma}>\frac{0-\mu}{\sigma}\right)=1-\Phi\left(-\frac{\mu}{\sigma}\right)=\Phi\left(\frac{\mu}{\sigma}\right)$

$\frac{\partial}{\partial\theta}P_\theta(X_1>0)=\left[\frac{1}{\sigma}\phi\left(\frac{\mu}{\sigma}\right)\quad -\frac{\mu}{\sigma^2}\phi\left(\frac{\mu}{\sigma}\right)\right]^t$

$\therefore RCLB:\left[\frac{1}{\sigma}\phi\left(\frac{\mu}{\sigma}\right)\quad -\frac{\mu}{\sigma^2}\phi\left(\frac{\mu}{\sigma}\right)\right]\begin{bmatrix}\sigma^2/n & 0\\ 0 & \frac{1}{2n}\sigma^2\end{bmatrix}\left[\frac{1}{\sigma}\phi\left(\frac{\mu}{\sigma}\right)\quad -\frac{\mu}{\sigma}\phi\left(\frac{\mu}{\sigma}\right)\right]^t$

$=\phi\left(\frac{\mu}{\sigma}\right)^2\cdot\frac{1}{n}+\frac{\mu^2}{\sigma^2}\phi\left(\frac{\mu}{\sigma}\right)^2\cdot\frac{1}{2n}=\phi\left(\frac{\mu}{\sigma}\right)^2\frac{1}{n}\left(1+\frac{\mu^2}{2\sigma^2}\right)$ ✓✓

$\sqrt{n}\left(\hat{\theta}^{MLE}-\theta\right)\xrightarrow[n\to\infty]{d}N(0,I(\theta)^{-1})\ \Rightarrow\ \sqrt{n}\left(g(\hat{\theta}^{MLE})-g(\theta)\right)\xrightarrow[n\to\infty]{d}N(0,\dot{g}(\theta)^tI(\theta)^{-1}\dot{g}(\theta))$

$\hat{\theta}^{MLE}=(\hat{\mu}^{MLE}\ \ \hat{\sigma}^{MLE})^t=\left(\bar{x}\ \ \sqrt{\frac{1}{n}\sum_{i=1}^n(x_i-\bar{x})^2}\right)^t=\left(\bar{x}\ \ \sqrt{\frac{n-1}{n}}s\right)^t$

$\eta=P_\theta(X>0)=P_\theta\left(\frac{X-\mu}{\sigma}>-\frac{\mu}{\sigma}\right)=1-\Phi\left(-\frac{\mu}{\sigma}\right)=\Phi\left(\frac{\mu}{\sigma}\right)$

$\frac{\partial}{\partial\theta}g(\theta)=\frac{\partial}{\partial\theta}\eta=\left[\frac{1}{\sigma}\phi\left(\frac{\mu}{\sigma}\right)\quad -\frac{\mu}{\sigma^2}\phi\left(\frac{\mu}{\sigma}\right)\right]^t$

$\sqrt{n}\left(g(\hat{\theta})-g(\theta)\right)\xrightarrow[n\to\infty]{d}N(0,\dot{g}(\theta)^tI(\theta)^{-1}\dot{g}(\theta))$

$\dot{g}(\theta)^tI(\theta)^{-1}\dot{g}(\theta)=\left[\frac{1}{\sigma}\phi\left(\frac{\mu}{\sigma}\right)\quad -\frac{\mu}{\sigma^2}\phi\left(\frac{\mu}{\sigma}\right)\right]\begin{bmatrix}\sigma^2 & 0\\ 0 & \frac{1}{2}\sigma^2\end{bmatrix}\left[\frac{1}{\sigma}\phi\left(\frac{\mu}{\sigma}\right)\quad -\frac{\mu}{\sigma^2}\phi\left(\frac{\mu}{\sigma}\right)\right]^t$

$=\phi\left(\frac{\mu}{\sigma}\right)^2+\frac{1}{2}\frac{\mu^2}{\sigma^2}\phi\left(\frac{\mu}{\sigma}\right)^2=\phi\left(\frac{\mu}{\sigma}\right)^2\left(1+\frac{\mu^2}{2\sigma^2}\right)$

$\therefore\sqrt{n}\left(\hat{\eta}^{MLE}-\eta\right)\longrightarrow N\left(0,\phi\left(\frac{\mu}{\sigma}\right)^2\left(1+\frac{\mu^2}{2\sigma^2}\right)\right)$ ✓✓

8.24. 확률밀도함수가

$$f(x;\alpha,\beta)=\alpha\beta^{-\alpha}x^{\alpha-1}\exp\left\{-x^\alpha/\beta^\alpha\right\}I_{(0,+\infty)}(x)$$

인 와이블분포 Weibull(α,β) , $\alpha>0$, $\beta>0$ 인 모형에서의 랜덤표본을 $X_1,\cdots,X_n$ 이라고 할 때, <정리 6.4.3>의 조건 (R0)~(R5)가 만족된다 것을 가정하고 $\log X_i$ $(i=1,2,\cdots,n)$의 적률을 이용한 추정량

$$\hat{\alpha}_n^0=\sqrt{1.6449/s^2_{\log x}}\ ,\quad s^2_{\log x}=\frac{1}{n}\sum_{i=1}^n(\log x_i-(\overline{\log x}))^2$$

의 최대가능도추정량 $\hat{\alpha}_n^{MLE}$에 대한 점근상대효율성을 구하여라. i.e., $ARE\left(\left\{\hat{\alpha}_n^0\right\},\left\{\hat{\alpha}^{MLE}\right\}\right)$

문제지 (6.19) 참고. $\hat{\alpha}^{r+1}=\hat{\alpha}^r+\left[-\ddot{g}(\hat{\alpha}^r)\right]^{-1}\dot{g}(\hat{\alpha}^r)$, $(r=0,1,\cdots)$

$\hat{\alpha}^0=\sqrt{1.6449/s^2_{\log x}}$, $s^2_{\log x}=\frac{1}{n}\sum_{i=1}^n(\log x_i-(\overline{\log x}))^2$

$\sqrt{n}(\hat{\alpha}^{MLE}-\alpha)\longrightarrow N(0,I(\theta)^{-1})$

$Var(\log X)=1.6449/\alpha^2\ \Leftrightarrow\ \alpha^2=1.6449/Var(\log X)$

$\Leftrightarrow\ \alpha=\sqrt{1.6449/Var(\log X)}$

$f(x;\alpha,\beta)=\alpha\beta^{-\alpha}x^{\alpha-1}\exp\left\{-x^\alpha/\beta^\alpha\right\}I_{(0,+\infty)}(x)$, $\theta=(\alpha,\beta)^t$

$\ell(\theta;x)=\log f(x;\theta)=\log\alpha-\alpha\log\beta+(\alpha-1)\log x-(x/\beta)^\alpha$

$\dot{\ell}_1(\theta)=\frac{\partial}{\partial\alpha}\ell(\theta;x)=\frac{1}{\alpha}-\log\beta+\log x-\log(x/\beta)\cdot(x/\beta)^\alpha$

$\dot{\ell}_2(\theta)=\frac{\partial}{\partial\beta}\ell(\theta;x)=-\frac{\alpha}{\beta}+\frac{\alpha x}{\beta^2}\left(\frac{x}{\beta}\right)^{\alpha-1}=-\frac{\alpha}{\beta}+\frac{\alpha}{\beta}\left(\frac{x}{\beta}\right)^\alpha$

$\ddot{\ell}_{11}(\theta)=\frac{\partial^2}{\partial\alpha^2}\ell(\theta;x)=-\alpha^{-2}-\log(x/\beta)^2(x/\beta)^\alpha$

$\ddot{\ell}_{12}(\theta)=\frac{\partial}{\partial\alpha\partial\beta}\ell(\theta;x)=-\frac{1}{\beta}-\frac{-x\beta^{-2}}{x\beta^{-1}}(x/\beta)^\alpha-\log(x/\beta)\cdot\frac{\alpha}{\beta}\left(\frac{x}{\beta}\right)^\alpha$

$=-\frac{1}{\beta}-\frac{1}{\beta}\left(\frac{x}{\beta}\right)^\alpha-\log\left(\frac{x}{\beta}\right)\frac{\alpha}{\beta}\left(\frac{x}{\beta}\right)^\alpha$

$\ddot{\ell}_{22}(\theta)=\frac{\alpha}{\beta^2}-\frac{\alpha}{\beta^2}\left(\frac{x}{\beta}\right)^\alpha-\frac{\alpha}{\beta}\cdot\alpha\frac{x}{\beta^2}\left(\frac{x}{\beta}\right)^{\alpha-1}=\frac{\alpha}{\beta^2}-\frac{\alpha}{\beta^2}\left(\frac{x}{\beta}\right)^\alpha-\frac{\alpha}{\beta^2}\left(\frac{x}{\beta}\right)^\alpha$

Let $Z=(X/\beta)^\alpha\sim Expl(1)$

$\ddot{\ell}_{11}(\theta)=-\alpha^{-2}-\alpha^{-2}\cdot\alpha^2\log(x/\beta)^2\left(\frac{x}{\beta}\right)^\alpha=-\alpha^{-2}-\alpha^{-2}\log(z)\cdot z$

$\ddot{\ell}_{12}(\theta)=-\beta^{-1}-\beta^{-1}z-\beta^{-1}z\log z$

$\ddot{\ell}_{22}(\theta)=\alpha\beta^{-2}-\alpha\beta^{-2}z-\alpha^2\beta^{-2}z=\alpha\beta^{-2}-\alpha\beta^{-2}(1+\alpha)z$

$$I(\theta)=E_\theta[-\ddot{\ell}(\theta)]=E\begin{bmatrix}\alpha^{-2}+\alpha^{-2}z\log z & \beta^{-1}+\beta^{-1}z+\beta^{-1}z\log z\\ \beta^{-1}+\beta^{-1}z+\beta^{-1}z\log z & -\alpha\beta^{-2}+\alpha\beta^{-2}(1+\alpha)z\end{bmatrix}$$

↑ 여기까지는 손으로 풀이 가능.

$\Psi(\alpha)=\frac{\partial}{\partial\alpha}\log\Gamma(\alpha)$, $\dot{\Psi}=\frac{\partial^2}{\partial\alpha^2}\log\Gamma(\alpha)$

↳ digamma ↳ Trigamma

$E[Z(\log z)^2]=\dot{\Psi}(1)+[\Psi(1)]^2+2\Psi(1)=0.8236$

$E[Z\log Z]=1+\Psi(1)=0.4228$

} 부분적분과 R을 이용

$$[I(\theta)]^{-1}=\begin{pmatrix}0.6079\alpha^2 & 0.2570\beta\\ 0.2570\beta & 1.1086\beta^2/\alpha^2\end{pmatrix}$$

따라서 α의 최대가능도 추정량의 극한분포는

$\sqrt{n}(\hat{\alpha}_n^{MLE}-\alpha)\xrightarrow[n\to\infty]{d}N(0,0.6079\alpha^2)$

($\hat{\alpha}^{(0)}$의 극한분포)

let $W=\log X$

$\sqrt{n}\left(\frac{1}{n}\sum_{i=1}^n(W_i-\bar{W})^2-Var(W)\right)\longrightarrow N(0,(\beta_4+2)\cdot Var(W)^2)$

$(X/\beta)^\alpha\sim Expl(1)$ $\quad P(X<x)=\exp\left\{-\left(\frac{x}{\beta}\right)^\alpha\right\}$

$P(\log X\le x)=P(X\le e^x)=\exp\left\{-e^{\alpha x}/\beta^\alpha\right\}$

$P(\log X=x)=-\frac{\alpha}{\beta^\alpha}e^{\alpha x}\exp\left\{-\left(\frac{e^x}{\beta}\right)^\alpha\right\}$

$\rho_4 = E[(W_i - E[W_i])^4] / Var(W_i)^2 - 3$

그런데 $a\log(X/\beta) = \log Z$, $Z \sim Exp(1)$ 이므로

$Var(W_i) = Var(\log Z)/a^2 = \psi(1)/a^2 = 1.6449/a^2$

$E[(W_i - E[W_i])^4] = E[(\log Z - E[\log Z])^4]/a^4$

한편, R의 psigamma(1,r) = $\psi^{(r)}(1)$ 을 이용하여 계산하면

$E[(\log Z - E[\log Z])^4] = T^{(4)}(1) - 4T^{(3)}(1) + 6T^{(2)}(1)[T^{(1)}(1)]^2 - 3[T^{(1)}(1)]^4$
$= 14.60260$

$\therefore \rho_4 = 14.60260/1.6449^2 = 2.396985$

$\therefore \sqrt{n}(S^2_{\log x} - 1.6449/a^2) \xrightarrow[n\to\infty]{d} N(0, (\rho_4+2)1.6449^2/a^4)$

$\therefore \sqrt{n}(\sqrt{1.6449/S^2_{\log x}} - a) \xrightarrow[n\to\infty]{d} N(0, (\rho_4+2)a^2/4)$

따라서 이 경우에 적률이용추정량의 최대가능도추정량에 대한 점근상대효율성은

$ARE(\{\hat{a}_n^0\}, \{\hat{a}_n^{MLE}\}) = [(\rho_4+2)a^2/4]^{-1} / [0.6079a^2]^{-1} = 0.5530$

★ 8.25. 〈예 8.3.8〉과 〈예 8.3.10〉에서 $\eta = \eta(\theta)$의 최대가능도추정량 $\hat{\eta}_n^{MLE}$ 과 전역최소분산 불편추정량 $\hat{\eta}_n^{UMVUE}$ 에 대하여

$$\sqrt{n}(\hat{\eta}_n^{UMVUE} - \hat{\eta}_n^{MLE}) \xrightarrow[n\to\infty]{P_\theta} 0$$

임을 밝히고, $\sqrt{n}(\hat{\eta}_n^{UMVUE} - \eta)$ 의 극한분포를 구하여라.

〈예 8.3.8〉

$X \sim Poisson(\theta)$ $\quad \hat{\eta}_n^{UMVUE} = (1-2/n)^{n\bar{X}} \simeq e^{-2\bar{X}} = \hat{\eta}_n^{MLE}$

* "converges in probability" : $X_n \xrightarrow[n\to\infty]{P_\theta} c \iff P(|X_n - c| > \epsilon) = 0 \quad \forall n \ge K(\epsilon)$

$\sqrt{n}(\hat{\eta}_n^{UMVUE} - \eta) - \sqrt{n}(\hat{\eta}_n^{MLE} - \eta) \xrightarrow[n\to\infty]{P_\theta} 0$

it is trivial that $\hat{\eta}_n^{UMVUE}, \hat{\eta}_n^{MLE} \xrightarrow[n\to\infty]{P_\theta} \eta$ ∵ UMVUE & MLE are unbiased

$\Rightarrow (\hat{\eta}_n^{UMVUE} - \hat{\eta}_n^{MLE}) \xrightarrow[n\to\infty]{P_\theta} 0$

WTS: $\forall \epsilon > 0$, $\exists K(\epsilon) \in \mathbb{N}$ s.t if $n \ge K(\epsilon)$, $P(|\sqrt{n}(\hat{\eta}_n^{UMVUE} - \hat{\eta}_n^{MLE})| > \epsilon) = 0$

How can we expand $(1-2/n)^{n\bar{X}}$? Binomial Expansion? Taylor series?

$$(1-\tfrac{2}{n})^{\Sigma X_i} = (1-m)^{\Sigma X_i} = f(m) = f(1) + \frac{f'(1)}{1!}(m-1) + \frac{f''(1)}{2!}(m-1)^2 + \cdots$$

$$e^{-2\bar{X}} = e^{-2\frac{1}{n}\Sigma X_i} = e^{-m\Sigma X_i} = g(m) = g(1) + \frac{g'(1)}{1!}(m-1) + \frac{g''(1)}{2!}(m-1)^2 + \cdots$$

Denote $\hat{\eta}_n^{UMVUE} = (1-\frac{2}{n})^{n\bar{X}}$ 을 간략하게 $\hat{\eta}_n$ 으로 나타내기로 한다.

$$-\log(\hat{\eta}_n) = -n\bar{X}\log(1-\tfrac{2}{n}) = n\bar{X}\left(\tfrac{2}{n} + \tfrac{1}{2}(\tfrac{2}{n})^2 + \tfrac{1}{3}(\tfrac{2}{n})^3 + \cdots\right)$$
$$= 2\bar{X} + n\bar{X}\left(\tfrac{1}{2}(\tfrac{2}{n})^2 + \tfrac{1}{3}(\tfrac{2}{n})^3 + \cdots\right)$$

* $-\log(1-A) = A + \frac{1}{2}A^2 + \frac{1}{3}A^3 + \cdots$ $= 2\bar{X} + R_n$, $\sqrt{n}R_n \xrightarrow[n\to 0]{P_\theta} 0$

$$\hat{\eta}_n = \exp\{\log(\hat{\eta}_n)\} = \exp\{-2\bar{X} - R_n\}$$
$$= \exp\{-2\bar{X}\} \cdot e^{-R_n} \xrightarrow[n\to\infty]{P_\theta} e^{-2\bar{X}} \cdot 1 \quad \because e^{-R_n} \to 0$$

∵ Slutsky's Theorem

(or) $= e^{-2\bar{X}}(1 - R_n + \frac{1}{2!}R_n^2 + \cdots) = e^{-2\bar{X}} + r_n$

$\sqrt{n}r_n = e^{-2\bar{X}}(-\sqrt{n}R_n) \xrightarrow[n\to\infty]{P_\theta} 0$

$$\therefore \sqrt{n}(\hat{\eta}_n - \hat{\eta}_n^{MLE}) = \sqrt{n}(\hat{\eta}_n - e^{-2\bar{X}}) \xrightarrow[n\to\infty]{P_\theta} 0$$

그런데 $\sqrt{n}(\hat{\eta}_n^{MLE} - e^{-2\theta}) = \sqrt{n}(e^{-2\bar{X}} - e^{-2\theta}) \xrightarrow[n\to\infty]{d} N(0, 4\theta e^{-4\theta})$

이므로 $\sqrt{n}(\hat{\eta}_n - e^{-2\theta}) = \sqrt{n}(\hat{\eta}_n - \hat{\eta}_n^{MLE}) + \sqrt{n}(\hat{\eta}_n^{MLE} - e^{-2\theta})$
$\xrightarrow[n\to\infty]{d} N(0, 4\theta e^{-4\theta})$

(b) 〈예 8.3.10〉

$X \sim Exp(\theta)$, $\eta = P_\theta(X_i > a)$, $\hat{\eta}_n^{UMVUE} = (1 - \frac{a/\bar{X}}{n})^{n-1} I_{(a/n,\infty)}(\bar{X}) \simeq e^{-a/\bar{X}} = \hat{\eta}_n^{MLE}$

$$\log(\hat{\eta}_n^{UMVUE}) = (n-1)\log(1-A) = -(n-1)\left\{A + \frac{1}{2}A^2 + \frac{1}{3}A^3 + \cdots\right\}$$
$$= -(n-1)\left\{\frac{a/\bar{X}}{n} + \frac{1}{2}\left[\frac{a/\bar{X}}{n}\right]^2 + \cdots\right\} = -(n-1)\frac{a/\bar{X}}{n} + R_n \text{ , } \sqrt{n}R_n \xrightarrow[n\to\infty]{P_\theta} 0$$

$$\hat{\eta}_n^{UMVUE} = \exp\{\log(\hat{\eta}_n^{UMVUE})\} = \exp\left\{-(n-1)\frac{a/\bar{X}}{n} + R_n\right\}$$

$$= \exp\left\{-(n-1)\frac{a/\bar{x}}{n} + R_n\right\} = \exp\left\{-a/\bar{x} + \frac{a/\bar{x}}{n} + R_n\right\}, \quad \sqrt{n}R_n \xrightarrow[n\to\infty]{P_\theta} 0$$

$$\sqrt{n}\exp\left\{-a/\bar{x} + \frac{a/\bar{x}}{n} + R_n\right\} = \sqrt{n}\exp\left\{-a/\bar{x}\right\}\exp\left\{\frac{a/\bar{x}}{n} + R_n\right\} \xrightarrow[n\to\infty]{P_\theta} \sqrt{n}\exp\left\{-a/\bar{x}\right\}$$

$$\therefore \sqrt{n}\left(\hat{\eta}^{UMVUE} - \hat{\eta}^{MLE}\right) \xrightarrow[n\to\infty]{P_\theta} \sqrt{n}\left\{\exp\left\{-a/\bar{x}\right\} - \exp\left\{-a/\bar{x}\right\}\right\} = 0$$

$$\sqrt{n}\left(\hat{\theta}^{MLE} - \theta\right) \longrightarrow N(0, I(\theta)^{-1})$$

$$f(x;\theta) = \frac{1}{\theta}e^{-x/\theta}, \quad \ell(\theta) = \log f(x;\theta) = -\log\theta - \frac{x}{\theta}$$

$$\ddot{\ell}(\theta) = \frac{\partial}{\partial\theta}\left\{-\frac{1}{\theta} + \frac{x}{\theta^2}\right\} = \frac{1}{\theta^2} - \frac{2x}{\theta^3}$$

$$E[-\ddot{\ell}(\theta)] = E\left[\frac{2x}{\theta^3} - \frac{1}{\theta^2}\right] = \frac{2}{\theta^2} - \frac{1}{\theta^2} = \theta^{-2} = I(\theta)$$

$$\therefore \sqrt{n}\left(\hat{\theta}^{MLE} - \theta\right) \xrightarrow[n\to\infty]{d} N(0, \theta^2)$$

$$\sqrt{n}\left(g(\hat{\theta}^{MLE}) - g(\theta)\right) \xrightarrow[n\to\infty]{d} N(0, \dot{g}(\theta)^2\theta^2)$$

$$g(\theta) = \exp\left\{-a/\theta\right\}, \quad \dot{g}(\theta) = \frac{a}{\theta^2}\exp\left\{-\frac{a}{\theta}\right\}, \quad \dot{g}(\theta)^2 = a^2\theta^{-4}\exp\left\{-\frac{2a}{\theta}\right\}$$

$$\therefore \sqrt{n}\left(\hat{\eta}_n^{UMVUE} - \eta\right) \xrightarrow[n\to\infty]{d} N(0, a^2\theta^{-2}\exp\left\{-\frac{2a}{\theta}\right\})$$

$$-n\log\left(1 - \frac{a/\bar{x}}{n}\right) = n\left[\frac{a/\bar{x}}{n} + \frac{1}{2}\left(\frac{a/\bar{x}}{n}\right)^2 + \cdots\right] = (a/\bar{x}) + \frac{1}{2}(a/\bar{x})^2\frac{1}{n} + \cdots$$

$$= (a/\bar{x}) + \frac{1}{2}\left[\sqrt{n}\left((a/\bar{x})^2 - (a/\theta)^2\right) + \sqrt{n}(a/\theta)^2\right]\frac{1}{n\sqrt{n}} + \cdots$$

$$= (a/\bar{x}) + R_n$$

$$\sqrt{n}R_n = \frac{1}{2}\left[\sqrt{n}\left((a/\bar{x})^2 - (a/\theta)^2\right) + \sqrt{n}(a/\theta)^2\right]\frac{1}{n} + \cdots \xrightarrow[n\to\infty]{P_\theta} 0$$

$$\left(1 - \frac{a/\bar{x}}{n}\right)^n = e^{-a/\bar{x}}\left(1 - R_n + \frac{1}{2!}R_n^2 + \cdots\right) = e^{-a/\bar{x}} + r_n$$

$$\sqrt{n}r_n = e^{-a/\bar{x}}\left(-\sqrt{n}R_n + \cdots\right) \xrightarrow[n\to\infty]{P_\theta} 0$$

또한 $\sqrt{n}\left|\left(1 - \frac{a/\bar{x}}{n}\right)^{n-1} - \left(1 - \frac{a/\bar{x}}{n}\right)^n\right| = \left|1 - \frac{a/\bar{x}}{n}\right)^{n-1}\frac{a/\bar{x}}{\sqrt{n}} \xrightarrow[n\to\infty]{P_\theta} 0$

$$\therefore \sqrt{n}\left|\left(1 - \frac{a/\bar{x}}{n}\right)^{n-1} - e^{-a/\bar{x}}\right| \xrightarrow[n\to\infty]{P_\theta} 0$$

왼쪽항 $P_\theta\left\{\sqrt{n}\left|1 - \frac{a/\bar{x}}{n}\right)^{n-1}\left(1 - I_{(a/n,\infty)}(\bar{x})\right| \geq \epsilon\right\} \leq P_\theta(\bar{x} \leq a/n)$

$$P_\theta(\bar{x} \leq a/n) \leq P_\theta(\bar{x} - \theta \leq -\theta + a/n \leq -\theta + \theta/2), \quad \forall n : a/n \leq \theta/2$$

$$0 \leq P_\theta(\bar{x} \leq a/n) \leq P_\theta(|\bar{x} - \theta| \geq \theta/2) \longrightarrow 0$$

$$\therefore \sqrt{n}\left(\hat{\eta}_n - \left(1 - \frac{a/\bar{x}}{n}\right)^{n-1}\right) \xrightarrow[n\to\infty]{P_\theta} 0$$

따라서 $\therefore \sqrt{n}\left(\hat{\eta}_n^{UMVUE} - \hat{\eta}_n^{MLE}\right) = \sqrt{n}\left(\hat{\eta}_n - e^{-a/\bar{x}}\right) \xrightarrow[n\to\infty]{P_\theta} 0$

Takeaway: How to expand estimators to polynomials:

1. use log transformation
2. $-\log(1-A) = A + \frac{1}{2}A^2 + \frac{1}{3}A^3 + \cdots$
3. use exp. transformation to revert step 1.