Ch.9 Comparison of Tests

9장 : 검정의 비교

9.1. CRITERIA FOR COMPARING TESTING PROCEDURES

To consider the easiest case for understanding the criteria used to compare testing procedures, let us examine the situation of testing a simple hypothesis where both the null hypothesis and the alternative hypothesis consist of a single pdf. That is, suppose the population distribution has a pdf $f(x;\theta)$, where $\theta \in \Omega = \{\theta_0, \theta_1\}$.

Using a random sample $X_1, \cdots, X_n$, consider testing the hypothesis

$$H_0 : \theta = \theta_0 \quad \text{vs} \quad H_1 : \theta = \theta_1$$

There are two types of probability of error in a test procedure with the critical region : "$X \in (X_1, \cdots, X_n)^t \in C$".

$$P_{\theta_0}(X \in C), \quad P_{\theta_1}(X \notin C) = 1 - P_{\theta_1}(X \in C)$$

In a randomized test $\phi(X) = \phi(X_1, \cdots, X_n)$, the probability of error is :

$$E_{\theta_0}\phi(X), \quad E_{\theta_1}[1-\phi(X)]$$

If there exists a test method that reduces both errors, it would be the best. However, if one error is reduced, the other increases and vice versa. Thus, it is typical to consider the following criterion, similar to what we have seen in chapter 8, p339.

Definition : 'Maximum Error Probability' and 'Bayesian Mean Error Probability'
'최대오류확률' '베이지안 평균오류확률'

(a) Criteria of Maximum Error Probability : Method of comparing tests $\phi(X)$ based on the maximum probability of error, that is:

$$\max\{E_{\theta_0}\phi(X), E_{\theta_1}[1-\phi(X)]\}$$

(b) Criteria of Bayesian Mean Error Probability : Method of comparing tests $\phi(X)$ based on the weighted average of error probabilities.

$$\pi_0 E_{\theta_0}\phi(X) + \pi_1 E_{\theta_1}[1-\phi(X)], \quad (\pi_0 + \pi_1 = 1, \ \pi_0 > 0, \ \pi_1 > 0)$$

We call π_0, π_1 prior probabilities.

〈Theorem 9.1.1 Bayes Test in Simple Hypothesis〉

Let the null and alternative hypotheses be:

$$H_0 : \theta = \theta_0 \quad \text{vs} \quad H_1 : \theta = \theta_1$$

The test that satisfies the following, minimizes the "Bayesian Mean Error Probability": 126)

$$\phi^{\pi}(x) = \begin{cases} 1 & pdf(x;\theta_1)/pdf(x;\theta_0) > \pi_0/\pi_1 \\ 0 & pdf(x;\theta_1)/pdf(x;\theta_0) < \pi_0/\pi_1 \end{cases}$$

(proof)

$$\pi_0 E_{\theta_0}\phi(X) + \pi_1 E_{\theta_1}[1-\phi(X)]$$

$$= \pi_0 \int \phi(x)\, pdf(x;\theta_0)\,dx + \pi_1 - \pi_1 \int \phi(x)\, pdf(x;\theta_1)\,dx$$

$$= \pi_0 \int \phi(x)\, pdf(x;\theta_0)\,dx - \pi_1 \int \phi(x) \frac{pdf(x;\theta_1)}{pdf(x;\theta_0)} \cdot pdf(x;\theta_0)\,dx$$

$$- \pi_1 \int \phi(x)\, pdf(x;\theta_1)\,dx \cdot I(pdf(x;\theta_0)=0) + \pi_1$$

$$= E_{\theta_0}\left[\pi_0\phi(X) - \pi_1\phi(X)\frac{pdf(x;\theta_1)}{pdf(x;\theta_0)}\right] - E_{\theta_1}[\pi_1\phi(X)\cdot I(pdf(x;\theta_0)=0)] + \pi_1$$

$$= -\pi_1\phi(X)\, E_{\theta_0}\left[\frac{pdf(x;\theta_1)}{pdf(x;\theta_0)} - \frac{\pi_0}{\pi_1}\right] - \pi_1 E_{\theta_1}[\phi(X) I(pdf(x;\theta_0)=0)] + \pi_1$$

Since we want to minimize the above equation,

if $\dfrac{pdf(x;\theta_1)}{pdf(x;\theta_0)} - \dfrac{\pi_0}{\pi_1} > 0 \Rightarrow \phi(X) = 1$

$* \ 0 \le \phi(X) \le 1$

if $\dfrac{pdf(x;\theta_1)}{pdf(x;\theta_0)} - \dfrac{\pi_0}{\pi_1} < 0 \Rightarrow \phi(X) = 0$

126) 여기에서 "양수÷0"은 $+\infty$로 어떤 실수보다도 더 큰 것으로 약속한다.

Ex. 9.1.1

Consider testing the simple hypothesis:

$$H_0 : \mu = \mu_0 \quad \text{vs} \quad H_1 : \mu = \mu_1$$

using the random samples drawn from $N(\mu, 1)$. The likelihood ratio is:

$$pdf(x;\mu_1)/pdf(x;\mu_0) = \exp\left\{n(\mu_1-\mu_0)\left(\bar{x} - \frac{\mu_1+\mu_0}{2}\right)\right\}$$

Thus, the test that minimizes the Bayesian Mean Error Probability is:

(a) when $\mu_1 > \mu_0$

$$\phi^{\pi}(x) = \begin{cases} 1, & (\bar{x}-\mu_0) > (\mu_1-\mu_0)/2 + \log(\pi_0/\pi_1)/(n(\mu_1-\mu_0)) \\ 0, & (\bar{x}-\mu_0) < (\mu_1-\mu_0)/2 + \log(\pi_0/\pi_1)/(n(\mu_1-\mu_0)) \end{cases}$$

(b) when $\mu_1 < \mu_0$

$$\phi^{\pi}(x) = \begin{cases} 1, & (\bar{x}-\mu_0) < (\mu_1-\mu_0)/2 + \log(\pi_0/\pi_1)/(n(\mu_1-\mu_0)) \\ 0, & (\bar{x}-\mu_0) > (\mu_1-\mu_0)/2 + \log(\pi_0/\pi_1)/(n(\mu_1-\mu_0)) \end{cases}$$

Comparing the tests using 'Bayesian Mean Error Probability' and 'Maximum Error Probability' implies that the null and alternative hypotheses are treated with equal importance. This differs from the formal logic (형식논리) of traditional tests covered in Chapter 7.[27]

Meanwhile, according to the formal logic of traditional tests, we compare tests whose significance level is α (Probability of falsely rejecting the null hypothesis is less than or equal to α). Thus, in these comparisons, the test that minimizes type II error $E_\theta[1-\phi(x)]$, or maximizes the power

$$\gamma_\phi(\theta_1) = E_{\theta_1}\phi(x)$$

is considered a good test.

[27] 베이지안 평균오류확률이나 최대오류확률을 기준으로 검정방법을 비교하는 것은 단순가설이 아닌 일반적인 가설의 경우에도 가능하지만 이 책의 수준을 넘는 것이므로 생략한다.

Definition: Most Powerful Test.[28]

In a simple hypothesis: $H_0: \theta = \theta_0$ vs $H_1: \theta = \theta_1$

Among the tests with significance level $\alpha\ (0<\alpha<1)$, the test that maximizes the power at $H_1: \theta = \theta_1$ is called the most powerful test with significance level α. Such test is denoted as $\phi^{MP\alpha}(x)$ or simply $\phi^{MP\alpha}$.

(i) (Significance level $\alpha\ (0<\alpha<1)$): $E_{\theta_0}\phi^{MP\alpha}(X) \le \alpha$

(ii) (Maximumum Power): $E_{\theta_1}\phi^{MP\alpha}(X) \ge E_{\theta_1}\phi(x) \quad \forall \phi: E_{\theta_0}[\phi(x)] \le \alpha$

< Theorem 9.1.2. Most Powerful Test in a simple hypothesis >

In a simple hypothesis:

$$H_0: \theta = \theta_0 \quad vs \quad H_1: \theta = \theta_1$$

(a) (Form of LRT)

$$\phi^*(x) = \begin{cases} 1, & pdf(x;\theta_1)/pdf(x;\theta_0) > K \\ \gamma, & pdf(x;\theta_1)/pdf(x;\theta_0) = K \\ 0, & pdf(x;\theta_1)/pdf(x;\theta_0) < K \end{cases}$$

(b) (Size of test α) $E_{\theta_0}\phi^*(x) = \alpha$

a test[29] that satisfies the above is a Most Powerful Test with Significance level α.

(proof) Similar to <Theorem 9.1.1>

$E_{\theta_1}\phi(x) - K E_{\theta_0}[\phi(x)]$

why start here? intuition connect with BME

* $E_{\theta_1}[\phi(x)]$ is the power, $E_{\theta_0}[\phi(x)]$ is the confidence level. We know that the two are inversely proportional. Thus, our objective is maximize $E_{\theta_1}[\phi(x)]$ and minimize $E_{\theta_0}[\phi(x)]$

* Bayesian Mean Error: $\pi_0 E_{\theta_0}[\phi(x)] + \pi_1 E_{\theta_1}[1-\phi(x)]$

$\downarrow \times(-1)$

$-\pi_0 E_{\theta_0}[\phi(x)] - \pi_1 + E_{\theta_1}[\phi(x)]$

$\therefore \min\ \pi_0 E_{\theta_0}[\phi(x)] + \pi_1 E_{\theta_1}[1-\phi(x)]$

$\Updownarrow$

$\max\ E_{\theta_1}[\phi(x)] - \pi_0 E_{\theta_0}[\phi(x)] - \pi_1$

$\Rightarrow \max E_{\theta}[\phi(x)] - K E_{\theta_0}[\phi(x)]$

$$E_{\theta_1}[\phi(x)] - K E_{\theta_0}[\phi(x)]$$

$$= E_{\theta_0}\left[\phi(x)\frac{pdf(x;\theta_1)}{pdf(x;\theta_0)}\right] + E_{\theta_1}[\phi(x) I(pdf(x;\theta_0)=0)] - K E_{\theta_0}[\phi(x)]$$

$$= E_{\theta_0}\left[\phi(x)\left\{\frac{pdf(x;\theta_1)}{pdf(x;\theta_0)} - K\right\}\right] + E_{\theta_1}[\phi(x) I(pdf(x;\theta_0)=0)]$$

Thus, from (a), for an arbitrary test $0 \le \phi(x) \le 1$, the following holds:

$$\left\{E_{\theta_1}[\phi^*(x)] - K E_{\theta_0}[\phi^*(x)]\right\} - \left\{E_{\theta_1}[\phi(x)] - K E_{\theta_0}[\phi(x)]\right\}$$

$$= E_{\theta_0}\left[(\phi^*(x)-\phi(x))\left\{\frac{pdf(x;\theta_1)}{pdf(x;\theta_0)} - K\right\}\right] + E_{\theta_1}[(\phi^*(x)-\phi(x)) I(pdf(x;\theta_0)=0)]$$

≥ 0

$= E_{\theta_1}[(1-\phi(x)) I(pdf(x;\theta_0)=0)]$

$\because \phi^*(x) > K$ when $pdf(x;\theta_0)=0$ due to Condition (a)

$$\therefore \left\{E_{\theta_1}[\phi^*(x)] - K E_{\theta_0}[\phi^*(x)]\right\} \ge \left\{E_{\theta_1}[\phi(x)] - K E_{\theta_0}[\phi(x)]\right\}$$

$$\Rightarrow \left\{E_{\theta_1}[\phi^*(x)] - E_{\theta_1}[\phi^*(x)]\right\} \ge K\left\{E_{\theta_0}[\phi^*(x)] - E_{\theta_0}[\phi(x)]\right\} = K(\alpha - E_{\theta_0}[\phi(x)]) \ge 0$$

(b) $\le \alpha$

[28] 조건 (a)에서 γ가 x에 의존해도 되고, 이러한 조건 (a)는 필요조건임도 알려져 있다. 즉 최강력검정은 (a)를 만족시켜야 하는 것도 알려져 있다.

[29] 일반적으로 (a), (b)를 만족시키는 K, γ가 존재하는 것이 알려져 있다.

Ex. 9.1.2.

Consider testing the simple hypothesis

$$H_0: \theta = 0.1 \quad vs \quad H_1: \theta = 0.05$$

using 100 random samples from Poisson(θ) distribution at significance level α. In this case, the likelihood ratio is:

$$pdf(x;\theta_1)/pdf(x;\theta_0) = e^{-n(\theta_1-\theta_0)}(\theta_1/\theta_0)^{x_1+\cdots+x_n} = e^5\left(\frac{1}{2}\right)^{x_1+\cdots+x_n} \quad (n=100)$$

The most powerful test with significance level α is:

$$\phi^*(x)=\begin{cases}1, & x_1+\cdots+x_n\le c-1\\ \gamma, & x_1+\cdots+x_n=c\\ 0, & x_1+\cdots+x_n\ge c+1\end{cases}\qquad (n=100),\ (c\ge 0,\ c\in N)$$

$$E_{\theta_0}\phi^*(x)=0.05\quad(\theta_0=0.1)$$

From <Ex 7.1.4>, we know that $c=5$ and $\gamma=\frac{21}{38}$

9.2. UNIFORMLY MOST POWERFUL TEST

Consider testing the general hypothesis:

$$H_0:\theta\in\Omega_0\quad vs\quad H_1:\theta\in\Omega_1\quad(\Omega_0\cap\Omega_1=\emptyset,\ \Omega_0\cup\Omega_1=\Omega)$$

at significance level α using random samples $X=(X_1,\cdots,X_n)^t$ drawn from a population distribution which is one of the pdf $f(x;\theta)$, $\theta\in\Omega$. The test that maximizes the power:

$$\gamma_\phi(\theta)=E_{\theta_1}\phi(x),\quad \theta_1\in\Omega$$

is a good test.

Definition: Uniformly Most Powerful Test

When testing a general hypothesis

$$H_0:\theta\in\Omega_0\quad vs\quad H_1:\theta\in\Omega_1$$

the test ϕ^{UMP_α} that satisfies the below are called uniformly most powerful test with significance level α.

(i) (Significance level α $(0\le\alpha\le1)$):

$$\max_{\theta\in\Omega_0}E_\theta\phi^{UMP_\alpha}(x)\le\alpha$$

(ii) (Maximum Power in Alternative Hypothesis)

$$E_{\theta_1}\phi^{UMP_\alpha}(x)\ge E_{\theta_1}\phi(x)\quad\forall\theta_1\in\Omega_1\quad\forall\phi:\max_{\theta\in\Omega_0}E_\theta\phi(x)\le\alpha$$

Ex 9.2.1

Consider testing the hypothesis:

$$H_0:\mu=\mu_0\quad vs\quad H_1:\mu>\mu_0$$

at significance level α $(0<\alpha<1)$ using random samples drawn from $N(\mu,1)$. Show that the UMP test is given as:

$$\phi^*(x)=\begin{cases}1, & \bar{x}-\mu\ge z_\alpha/\sqrt{n}\\ 0, & \bar{x}-\mu<z_\alpha/\sqrt{n}\end{cases}$$

(Solution) The most powerful test for testing the simple hypothesis:

$$H_0:\mu=\mu_0\quad vs\quad H_1:\mu=\mu_1$$

at significance level α is given in <Theorem 9.1.2>. However, the likelihood ratio is given as:

$$pdf(x;\mu_1)/pdf(x;\mu_0)=\exp\left\{n(\mu_1-\mu_0)\left(\bar{x}-\frac{\mu_1+\mu_0}{2}\right)\right\}$$

and it is an increasing function of $\bar{x}$. Hence, the most powerful test is:

$$\phi^*(x) = \begin{cases} 1 & \bar{x} \geq c \\ 0 & \bar{x} < c \end{cases}, \quad E_{\mu_0}\phi^*(x) = \alpha$$

Under $H_0: \mu = \mu_0$, $\bar{x} \sim N(\mu_0, 1/n)$. Thus, $c = \mu_0 + z_\alpha/\sqrt{n}$ and the most powerful test does not depend on the value of μ_1. Therefore, this test is an UMP test with significance level α.

key Takeaways:

1. Start with a Simple hypothesis
2. Find C using $\bar{x}$ (statistic that influences the Likelihood Ratio)
3. Check if μ_1 (value of H_1) affects the Likelihood Ratio
   - ↳ No: Stop ⟶ $\phi(x)$ is a UMP Test
   - ↳ Yes: Find μ_1 that maximizes the power.

Ex. 9.2.2.

Consider testing the hypothesis

$$H_0(-): \mu \leq \mu_0 \quad vs \quad H_1: \mu > \mu_1$$

at Significance level α using the random samples drawn from $N(\mu, 1)$

(Solution)

First, for the test in ⟨Ex. 9.2.1⟩, we know that its power function is:

$$\begin{aligned} E_\mu\phi^*(x) &= P_\mu(\bar{x} - \mu_0 \geq z_\alpha/\sqrt{n}) = P_\mu(\sqrt{n}(\bar{x} - \mu_0) \geq z_\alpha) \\ &= P_\mu(\sqrt{n}(\bar{x} - \mu) - \sqrt{n}(\mu_0 - \mu) \geq z_\alpha) \\ &= P_\mu(\sqrt{n}(\bar{x} - \mu) \geq \sqrt{n}(\mu_0 - \mu) + z_\alpha) \\ &= P_\mu(Z \geq \sqrt{n}(\mu_0 - \mu) + z_\alpha), \quad Z \sim N(0,1) \end{aligned}$$

which is an increasing function of μ. Hence,

$$\max_{\mu \leq \mu_0} E_\mu(\phi^*(x)) = E_{\mu_0}[\phi^*(x)] = \alpha$$

Meaning, the test ϕ^* in ⟨Ex. 9.2.1⟩ has significance level α even if $H_0(-): \mu \leq \mu_0$.

Second, a significance level α test for $H_0(-): \mu \leq \mu_0$ is also a significance level α test for $H_0: \mu = \mu_0$. Meaning:

$$\{\phi : \max_{\mu \leq \mu_0} E_\mu\phi(x) \leq \alpha, \ 0 \leq \phi(x) \leq 1\} \subseteq \{\phi : E_{\mu_0}[\phi(x)] \leq \alpha, \ 0 \leq \phi(x) \leq 1\}$$

(Stronger constraint)

$$E_{\mu_0}[\phi(x)] \leq \max_{\mu \leq \mu_0} E[\phi(x)] \leq \alpha$$

However, the test ϕ^* from ⟨Ex. 9.2.1⟩ is a test with the maximum power among the tests in RHS (*). Thus, naturally ϕ^* has a greater or equal power than the tests in the LHS.

* RHS is a bigger & inclusive set of the LHS set.
ϕ^* = max power test in RHS $\{\phi\}$ ∴ ϕ^* is UMP

Hence the test ϕ^* from ⟨Ex. 9.2.1⟩ is also a UMP test for testing

$$H_0(-): \mu \leq \mu_0 \quad vs \quad H_1: \mu > \mu_0$$

at Significance level α.[130]

⟨Theorem 9.2.1 Single Parameter Exponential Family and UMP Tests⟩

If the pdf of the population distribution is:

$$f(x;\theta) = \exp\{g(\theta)T(x) - B(\theta) + S(x)\}, \quad x \in \mathcal{X}, \ \theta \in \Omega \subseteq \mathbb{R}^1$$

and $g(\theta)$ is an increasing function of θ, consider testing the hypothesis

$$H_0: \theta \leq \theta_0 \quad vs. \quad H_1: \theta > \theta_0$$

at significance level α $(0 < \alpha < 1)$. The test ϕ^* that satisfies the below is the significance level α UMP test.

(a) (Likelihood Ratio Shape) $\phi^*(x) = \begin{cases} 1, & T(x_1) + \cdots + T(x_n) > c \\ \gamma, & T(x_1) + \cdots + T(x_n) = c \\ 0, & T(x_1) + \cdots + T(x_n) < c \end{cases}$

(b) (Size of test α) $E_{\theta_0}\phi^*(x) = \alpha$

(proof) The Significance level α MP (most powerful) test for testing:

$$H_0(\theta_0): \theta = \theta_0 \quad vs \quad H_1(\theta_1): \theta = \theta_1 \quad (\theta_1 > \theta_0)$$

can be calculated using ⟨Theorem 9.1.2⟩. However, the likelihood ratio is

$$\begin{aligned} &pdf(x;\theta_1) / pdf(x;\theta_0) \\ &= \exp\{\underbrace{(g(\theta_1) - g(\theta_0))}_{g(\theta) \text{ is an increasing function of } \theta,\ \theta_1 > \theta_0 \ \therefore g(\theta_1) - g(\theta_0) > 0}(T(x_1) + \cdots + T(x_n)) - B(\theta_1) + B(\theta_0)\} \end{aligned}$$

which is an increasing function of $T(x_1) + \cdots + T(x_n)$. Thus, the MP test is determined by (a) and (b).

130) 귀무가설과 대립가설이 $H_0: \theta \geq \theta_0$ vs $H_1: \theta < \theta_0$ 인 경우에 유의수준 α의 전역최강력검정을 위한 조건은 다음과 같다.

(a) $\phi^*(x) = \begin{cases} 1, & T(x_1) + \cdots + T(x_n) < c \\ \gamma, & T(x_1) + \cdots + T(x_n) = c \\ 0, & T(x_1) + \cdots + T(x_n) > c \end{cases}$ (b) $E_{\theta_0}[\phi^*(x)] = \alpha$

In addition, such MP test is not dependent on the value of θ_1. Thus $\phi^*(x)$ is an UMP test with Significance level α for testing the hypothesis:

$$H_0(\theta_0): \theta = \theta_0 \quad vs \quad H_1: \theta > \theta_0$$

(continued on the next page)

Secondly, Similar to ⟨Ex.9.2.2⟩, if we can show that the power function for ϕ^*, $E_\theta[\phi^*]$, is an increasing function of θ, the following holds

$$\max_{\theta \le \theta_0} E_\theta[\phi^*(X)] = E_{\theta_0}[\phi^*(X)] = \alpha$$

and thus the test ϕ^* is a Significance level α test even if $H_0: \theta \le \theta_0$. Furthermore, we can utilize:

$$\left\{\phi : \max_{\theta \le \theta_0} E_\theta[\phi(x)] \le \alpha,\ 0 \le \phi(x) \le 1\right\} \subseteq \left\{\phi : E_{\theta_0}[\phi(x)] \le \alpha,\ 0 \le \phi(x) \le 1\right\}$$

to argue that the test ϕ^* is a UMP test with significance level α $(0<\alpha<1)$ for testing

$$H_0: \theta \le \theta_0 \quad vs \quad H_1: \theta > \theta_0$$

We can show that the power function for test ϕ^*, $E_\theta[\phi^*(X)]$ is an increasing function of θ in the following manner:

Let θ' and θ'' $(\theta'' > \theta')$ be arbitrary parameters in the parameter space and let $\alpha' = E_{\theta'}\phi^*(X)$.

Consider testing the hypothesis:

$$H_0(\theta'): \theta = \theta' \quad vs \quad H_1(\theta''): \theta = \theta''$$

at Significance level α'.

Let $\phi_{\alpha'}(x) \overset{x}{\equiv} \alpha'$ be a randomized test that always rejects the H_0 with probability α'. Then, it is trivial that its power is less than or equal to the power of the most powerful test $\phi^{MP_{\alpha'}}$. I.e.,

$$E_{\theta''}[\phi^{MP_{\alpha'}}(X)] \ge E_{\theta''}[\phi_{\alpha'}(X)] = \alpha'$$

From ⟨Theorem 9.1.2⟩, we know that MP test $\phi^{MP_{\alpha'}}$ is determined by (a) and (b): $E_{\theta'}\phi^*(x) = \alpha'$. Thus, $\phi^{MP_{\alpha'}}$ and ϕ^* are equivalent. Therefore,

$$E_{\theta''}[\phi^*(X)] \ge E_{\theta''}\phi_{\alpha'}(x) = \alpha' = E_{\theta'}[\phi^*(X)]$$

$\therefore E_\theta[\phi^*(X)]$ is an increasing function of θ. □

if $\theta'' < \theta'$ $E_\theta[\phi^*(x)]$ is a decreasing function of θ : $\theta \downarrow$ and $E_\theta[\phi(x)] \uparrow$

Steps of proof

1. start from Simple hypothesis $H_0: \theta = \theta_0$ vs $H_1: \theta = \theta_1$
   Find the MP test
2. Show that test $\phi(x)$ does not depend on θ_1
   $\therefore$ MP test in step 1 is valid for the hypothesis $H_0: \theta = \theta_0$ vs. $H_1: \theta > \theta_0$
3. Show that the power function of the test : $E_\theta[\phi(X)]$ is an increasing function of θ
4. Conclusion : MP test in (Step 1) $\phi^*(x)$ is a UMP test for testing $H_0: \theta \le \theta_0$ vs. $H_1: \theta > \theta_0$

Ex. 9.2.3. One-Sided test for mean of Poisson distribution.

Find the UMP test with significance level $\alpha = 0.05$ for testing:

$$H_0: \theta \ge 0.1 \quad vs \quad H_1: \theta < 0.1$$

using $n=100$ random samples $X_1, \cdots, X_n$ drawn from Poisson(θ)

$$f(x;\theta) = \frac{e^{-\theta}\theta^x}{x!} = \exp\{-\theta + x\log\theta - \log x!\}$$

$T(x) = X$ and $g(\theta) = \log\theta$. $g(\theta)$ is an increasing function of θ.

$$\phi^*(x) = \begin{cases} 1, & \sum X_i < C \\ \gamma, & \sum X_i = C \\ 0, & \sum X_i > C \end{cases} \Leftrightarrow \phi^*(x) = \begin{cases} 1 & \sum X_i \le C-1 \\ \gamma & \sum X_i = C \\ 0 & \sum X_i \ge C+1 \end{cases}$$

$$E_{\theta_0}[\phi^*(X)] = P(\textstyle\sum X_i \le C-1) + \gamma P(\sum X_i = C) = 0.05,\ 0<\gamma<1$$

Under H_0: $\sum X_i \sim$ Poisson$(0.1\times 100) \overset{d}{\equiv}$ Poisson(10)

$$\therefore E_{\theta_0}[\phi^*(X)] = P(\textstyle\sum X_i \le 4) + \gamma P(\sum X_i = 5) = 0.05$$
$$= 0.02925 + \gamma \cdot 0.03783 = 0.05$$

$$\therefore \gamma = (0.05 - 0.02925) \div 0.03783 =$$

$\therefore C = 5$, $\gamma = 0.53552\ldots$ ✓✓

Ex. 9.2.4. one-sided test in mean of Exp distribution

Find the UMP test with significance level α $(0<\alpha<1)$ for testing:

$$H_0: \theta \le \theta_0 \quad vs \quad H_1: \theta > \theta_0 \quad (\theta_0 \text{ is given})$$

using random samples $X_1, \cdots, X_n$ drawn from Exp(θ)

$$f(x;\theta) = \frac{1}{\theta}\exp\left\{-\frac{x}{\theta}\right\} = \exp\left\{-\log\theta - \frac{1}{\theta}\cdot x\right\}$$

$g(\theta) = -\frac{1}{\theta}$, $\therefore g(\theta)$ is an increasing function of θ

$T(x) = x$

$$\phi^*(x) = \begin{cases} 1, & X_1 + \cdots + X_n > c \\ \gamma, & X_1 + \cdots + X_n = c \\ 0, & X_1 + \cdots + X_n < c \end{cases} \Rightarrow \phi^*(x) \begin{cases} 1, & X_1 + \cdots + X_n \ge c \\ 0, & X_1 + \cdots + X_n < c \end{cases}$$

$$E_{\theta_0}[\phi^*(X)] = \alpha$$

$$\frac{X_i}{\theta} \sim Exp(1) \overset{d}{\equiv} Gamma(1,1) \quad \therefore \frac{\sum X_i}{\theta} \sim Gamma(n,1) \Rightarrow \frac{2\sum X_i}{\theta} \sim Gamma(n,2) \overset{d}{\equiv} \chi^2(2n)$$

$$E_{\theta_0}[\phi^*(X)] = P_{\theta_0}(\textstyle\sum X_n > c) = P(\frac{2\sum X_n}{\theta} > c') = \alpha$$

$$\therefore c' = \chi^2_{1-\alpha}(2n) \quad \therefore c = \frac{\theta}{2}\cdot\chi^2_{1-\alpha}(2n)$$

$\chi^2_\alpha(2n)$ $\therefore c = \frac{\theta}{2}\chi^2_\alpha(2n)$

notation 주의!!

☆ 예 9.2.5. two-sided test for mean of normal distribution

Show that a UMP test with significance level α does not exist for testing

$$H_0: \mu = \mu_0 \quad \text{vs} \quad H_1: \mu \neq \mu_0$$

using random samples drawn from $N(\mu, 1)$, $-\infty < \mu < +\infty$.

$$f(x;\theta) = (2\pi)^{-1/2} \exp\left\{-\tfrac{1}{2}(x-\mu)^2\right\}$$
$$= \exp\left\{-\tfrac{1}{2}\log 2\pi - \tfrac{1}{2}(x^2 - 2\mu x + \mu^2)\right\}$$

$g(\mu) = \mu$: increasing function of μ

$T(x) = x$

if $\mu_0 \geq \mu_1$

$$\phi^*(x) = \begin{cases} 1, & X_1 + \cdots + X_n \leq c \\ 0, & X_1 + \cdots + X_n > c \end{cases}$$

if $\mu_0 < \mu_1$

$$\phi^*(x) = \begin{cases} 1, & X_1 + \cdots + X_n \geq c \\ 0, & X_1 + \cdots + X_n < c \end{cases}$$

no such $\phi^*(x)$ exists

[Textbook Approach]

〈정리 9.1.2〉의 필요조건으로부터 다음 두 조건을 모두 만족시키는 C_1, C_2가 존재하여야 한다.

$$\phi^*(x) = \begin{cases} 1, & X_1 + \cdots + X_n \leq C_1 \\ 0, & X_1 + \cdots + X_n > C_1 \end{cases} \qquad \phi^*(x) = \begin{cases} 1, & X_1 + \cdots + X_n \geq C_2 \\ 0, & X_1 + \cdots + X_n < C_2 \end{cases}$$

이러한 상수 C_1, C_2는 존재하지 않으므로 전역최강력 검정은 존재하지 않는다.

p. 390 참고

☆ 예 9.2.6 Properties of the two-sided test for mean of Normal Distributions.[131]

Consider the maximum LRT

$$\phi^*(x) = \begin{cases} 1 & |\sqrt{n}(\bar{x} - \theta_0)| \geq z_{\alpha/2} \\ 0 & |\sqrt{n}(\bar{x} - \theta_0)| < z_{\alpha/2} \end{cases}$$

which tests the hypothesis:

$$H_0: \theta = \theta_0 \quad \text{vs} \quad H_1: \theta \neq \theta_0$$

at significance level α $(0 < \alpha < 1)$ using random samples $X = (X_1, \cdots, X_n)^t$ drawn from $N(\theta, 1)$, $-\infty < \theta < +\infty$. Show that the following are true for the maximum LRT.

(a) $E_{\theta_0}[\phi^*(x)] = \alpha$, $\left[\frac{d}{d\theta} E_\theta[\phi^*(x)]\right]_{\theta=\theta_0} = 0$

(b) $\forall \phi: E_{\theta_0}[\phi(x)] = \alpha$, $\left[\frac{d}{d\theta} E_\theta[\phi(\theta)]\right]_{\theta=\theta_0} = 0$,

$$E_{\theta_1}[\phi^*(\theta)] \geq E_{\theta_1}[\phi(x)] \quad \forall \theta_1 : \theta_1 \neq \theta_0$$

(proof)

(a) $E_{\theta_0}[\phi^*(x)] = P_{\theta_0}(|\sqrt{n}(\bar{x} - \theta_0)| > z_{\alpha/2})$

$= P_{\theta_0}(|Z| > z_{\alpha/2}) = \alpha \quad Z \sim N(0,1)$

$= 1 - P_{\theta_0}(|Z| < z_{\alpha/2})$

$= 1 - (\Phi(z_{\alpha/2}) - \Phi(-z_{\alpha/2}))$

$= 1 - (1 - \alpha/2 - \alpha/2)$

$= \alpha$

$E_\theta[\phi^*(x)] = P_\theta(|\sqrt{n}(\bar{x} - \theta_0)| > z_{\alpha/2})$

$= 1 - P_\theta(|\sqrt{n}(\bar{x} - \theta + \theta - \theta_0)| < z_{\alpha/2})$

$= 1 - P_\theta(-z_{\alpha/2} < \sqrt{n}(\bar{x} - \theta) + \sqrt{n}(\theta - \theta_0) < z_{\alpha/2})$

$= 1 - P_\theta(-z_{\alpha/2} - \delta < \sqrt{n}(\bar{x} - \theta) < z_{\alpha/2} - \delta), \quad \delta = \sqrt{n}(\theta - \theta_0)$

$= 1 - (\Phi(z_{\alpha/2} - \delta) - \Phi(-z_{\alpha/2} - \delta))$

$= 1 - (\Phi(z_{\alpha/2} - \sqrt{n}(\theta - \theta_0)) - \Phi(-z_{\alpha/2} - \sqrt{n}(\theta - \theta_0)))$

$$\frac{\partial}{\partial\theta} E_\theta[\phi^*(x)] = \sqrt{n}\,\Phi(z_{\alpha/2} - \sqrt{n}(\theta - \theta_0)) + \sqrt{n}\,\Phi(-z_{\alpha/2} - \sqrt{n}(\theta - \theta_0))$$

$$\left[\frac{\partial}{\partial\theta} E_\theta[\phi^*(x)]\right]_{\theta=\theta_0} = \sqrt{n}\,\Phi(z_{\alpha/2}) + \sqrt{n}\,\Phi(-z_{\alpha/2}) = \sqrt{n}\,\Phi(z_{\alpha/2}) - \sqrt{n}\,\Phi(z_{\alpha/2})$$
$$= 0$$

(b) Let $p_\theta(x) = pdf(x;\theta)$, $\dot{p}_\theta(x) = \frac{\partial}{\partial\theta} pdf(x;\theta)$

$$\left[\frac{d}{d\theta} E_\theta[\phi(x)]\right]_{\theta=\theta_0} = \left[\frac{\partial}{\partial\theta} \int \phi(x) \cdot p_\theta(x)\,dx\right]_{\theta=\theta_0}$$

$$= \left[\int \phi(x) \cdot \dot{p}_\theta(x)\,dx\right]_{\theta=\theta_0} = \left[\int \phi(x) \frac{\dot{p}_\theta(x)}{p_\theta(x)} \cdot p_\theta(x)\,dx\right]_{\theta=\theta_0}$$

$$= \left[E_\theta\left[\phi(x) \frac{\dot{p}_\theta(x)}{p_\theta(x)}\right]\right]_{\theta=\theta_0} = E_{\theta_0}\left[\phi(x) \frac{\dot{p}_{\theta_0}(x)}{p_{\theta_0}(x)}\right]$$

[131] 일반적으로 조건 $\max_{\theta \in \Omega_0} E_\theta[\phi(x)] \leq \alpha$, $\min_{\theta \in \Omega_1} E_\theta[\phi(x)] \geq \alpha$ 를 만족시키는 검정 $\phi(x)$를 유의수준 α의 치우침 없는 검정 또는 불편검정(unbiased test)이라고 한다. 또한 이러한 조건을 만족시키며 대립가설 하의 검정력을 가장 크게 하는 검정을 유의수준 α의 전역최강력 불편검정(uniformly most powerful unbiased test)이라고 한다. 〈예 9.2.6〉에서 조건 (a)는 이러한 불편성의 조건을 대입한 것으로서 (b)로부터 이 예에서의 양쪽검정은 전역최강력 불편검정임을 알 수 있다. 이러한 전역최강력 불편검정은 단일모수지수족의 경우는 물론 다중모수 지수족의 경우에도 유사한 방법으로 찾을 수 있다고 알려져 있다. 이에 대한 상세한 논의는 [E.L. Lehmann & J.P. Romano] Testing Statistical Hypothesis 3rd ed. 에 주어져 있다.

Secondly, consider $E_{\theta_1}[\phi(x)] - K_1 E_{\theta_0}[\phi(x)] - K_2 E_{\theta_0}[\phi(x)\dot{p}_{\theta_0}(x)/p_{\theta_0}(x)]$ where K_1 and K_2 are multipliers (승수) and $\theta_1 \neq \theta_0$, Similar to 〈Thm. 9.1.2〉.

$$E_{\theta_1}[\phi(x)] - K_1 E_{\theta_0}[\phi(x)] - K_2 E_{\theta_0}[\phi(x)\dot{p}_{\theta_0}(x)/p_{\theta_0}(x)]$$

$$= E_{\theta_0}\left[\phi(x)\left\{p_{\theta_1}(x)/p_{\theta_0}(x) - K_1 - K_2 \dot{p}_{\theta_1}(x)/p_{\theta_0}(x)\right\}\right]$$

Test ϕ^{**} that maximizes the above is:

$$\phi^{**} = \begin{cases} 1, & p_{\theta_1}(x)/p_{\theta_0}(x) \geq K_1 + K_2 \dot{p}_{\theta_1}(x)/p_{\theta_0}(x) \\ 0, & p_{\theta_1}(x)/p_{\theta_0}(x) < K_1 + K_2 \dot{p}_{\theta_1}(x)/p_{\theta_0}(x) \end{cases}$$

Therefore,

$$E_{\theta_1}[\phi^{**}(X)] - K_1 E_{\theta_0}[\phi^{**}(X)] - K_2 E_{\theta_0}[\phi^{**}(X)\, p(\theta;x)/p(\theta;x)]$$
$$\geq E_{\theta_1}[\phi(X)] - K_1 E_{\theta_0}[\phi(X)] - K_2 E_{\theta_0}[\phi(x)\, p(\theta;x)/p(\theta;x)]$$

$$\Rightarrow E_{\theta_1}[\phi^{**}(x)] - E_{\theta_1}[\phi(x)] \geq K_1\{E_{\theta_0}[\phi^{**}(x)] - E_{\theta_0}[\phi(x)]\}$$
$$+ K_2\{E_{\theta_0}[\phi^{**}(x)\, p(\theta;x)/p(\theta;x)] - E_{\theta_0}[\phi(x)\, p(\theta;x)/p(\theta;x)]\}$$

If we can find positive real values K_1, K_2 which allows test ϕ^{**} to satisfy condition (a), the following holds:

$$E_{\theta_1}\phi^{**}(x) - E_{\theta_1}\phi(x) \geq 0 \quad \forall \phi : E_{\theta_0}[\phi(x)] = a,\ \left[\frac{d}{d\theta}E_\theta\phi(x)\right]_{\theta=\theta_0} = 0$$

$$\therefore E_{\theta_1}[\phi^{**}] \geq E_{\theta_1}[\phi(x)]$$

Lastly, if we can find K_1 and K_2 that satisfies (b), we can show that ϕ^{**} and ϕ^* are equivalent. ★ p.388 마지막 문단 참고

$$pdf(x;\theta_1)/pdf(x;\theta_0) \simeq log\left(\frac{\hat{\sigma}_0^2}{\hat{\sigma}_1^2}\right) = log\left(\frac{\sum(x_i-\theta_0)^2}{\sum(x_i-\bar{x})^2}\right) = log\left(1+\frac{n(\bar{x}-\theta_0)^2}{\hat{\sigma}^2}\right)$$
↳ $\frac{1}{2}$ was omitted for simplicity

$$\phi(x) = \begin{cases} 1 & \text{if } \dfrac{(\bar{x}-\theta_0)^2}{S_n/\sqrt{n}} > c \\ 0 & \text{if } \dfrac{(\bar{x}-\theta_0)^2}{S_n/\sqrt{n}} \leq c \end{cases}$$

$$\frac{(\bar{x}-\theta_0)^2}{S_n/\sqrt{n}} \sim t(n-1)$$

$$\therefore \text{test} : ① \ \frac{\bar{x}-\theta_0}{S_n/\sqrt{n}} \geq a \quad ② \ \frac{\bar{x}-\theta_0}{S_n/\sqrt{n}} \leq b \quad ③ \left|\frac{\bar{x}-\theta_0}{S_n/\sqrt{n}}\right| \geq c$$

$$\left[\frac{d}{d\theta}E_\theta[\phi(x)]\right]_{\theta=\theta_0} = 0$$

∴ Test shape is ③ since the other shape/form's power function depends on θ.

(…)

9.3. NONPARAMETRIC TEST AND ASYMPTOTIC COMPARISON

Let us consider hypothesis tests in situations where, in specifying the model for the population distribution, we do not assume a particular functional form of the probability density function, such as the normal distribution or the logistic distribution.

Ex. 9.3.1 Sign test for a location parameter model.

Similar to ⟨Ex. 8.4.3⟩, consider a model whose population distribution is continuous, its pdf is :

$$f(x-\theta),\ -\infty < \theta < +\infty$$

and is symmetric about θ, i.e. $f(-x) = f(x)\ \forall x : -\infty < x < +\infty$ and its cdf F is strictly increasing.

Consider testing the hypothesis:

$$H_0(\theta_0) : \theta = \theta_0 \quad vs \quad H_1 : \theta > \theta_0$$

at significance level $a\ (0 < a < 1)$ using the statistic

$$S_n = \sum_{i=1}^{n} I(x_i > \theta_0)$$

Here, $X_1, \cdots, X_n$ are drawn from the model mentioned previously.

In this case, we know that

* $I(x_i > \theta_0) \sim Bernoulli(p(\theta))$
∴ $\sum I(x_i > \theta_0) \sim Bin(n, p(\theta))$

$$S_n \sim B(n, p(\theta)),\ p(\theta) = P_\theta(X_1 > \theta_0) = 1 - F(\theta_0 - \theta)$$
↳ Binomial

and the hypothesis above corresponds to a hypothesis for $p(\theta)$:

$$H_0(1/2) : \underbrace{p(\theta) = 1/2}_{(1)} \quad vs \quad H_1 : \underbrace{p(\theta) > 1/2}_{(2)}$$

(1) if $H_0 : \theta = \theta_0$ is true, $p(\theta) = 1 - F(\theta_0 - \theta) = 1 - F(0) = 1 - 1/2$

(2) if $H_1 : \theta > \theta_0$ is true, $\theta_0 - \theta < 0$ ∴ $F(\theta_0 - \theta) < 1/2 \Leftrightarrow 1 - F(\theta_0 - \theta) > 1/2 \Leftrightarrow p(\theta) > 1/2$

Thus, we can use the following test as a significance level a test:

$$\phi_S(X_1, \cdots, X_n) = \begin{cases} 1, & S_n \geq c+1 \\ \gamma, & S_n = c \\ 0, & S_n \leq c-1 \end{cases} \quad (0 \leq \gamma \leq 1)$$

∵ Binomial dist is a member of the exponential family

$$E_{\theta_0}[\phi_S(X)] = P_{\theta_0}(S_n \geq c+1) + \gamma P_{\theta_0}(S_n = c) = a,$$

Meaning, if $\sum_{k=c+1}^{n} \binom{n}{k}(1/2)^n + \gamma\binom{n}{c}(1/2)^n = a$,

$$E_\theta[\phi_S(X_1, \cdots, X_n)] = E[\phi_S(Z_1+\theta, \cdots, Z_n+\theta)],\ Z_i \overset{iid}{\sim} f\ (i = 1, 2, \cdots, n)$$

and because $\phi_S(x_1, \cdots, x_n)$ is an increasing function for each component $x_i\ (i = 1, \cdots, n)$, the following holds:

$$\max_{\theta \le \theta_0} E_\theta \phi_S(X) = E_{\theta_0} \phi_S(X) = \alpha$$

Hence, the test ϕ_S can be used as a significance level α test for testing the hypothesis:

$$H_0 : \theta \le \theta_0 \quad \text{vs.} \quad H_1 : \theta > \theta_0$$

As such, a method of testing the median of a continuous distribution using the statistic S_n is referred to as the "sign test (부호검정)".

Ex. 9.3.2. Approximation of the Power of the sign test in a Location parameter model

For the hypothesis in ⟨Ex. 9.3.1⟩ :

$$H_0(\theta_0) : \theta = \theta_0 \quad \text{vs} \quad H_1 : \theta > \theta_0$$

Apply the normal approximation of Binomial distribution to the asymptotic distribution of S_n, a sign test statistic, and we get :

$$\frac{S_n - np(\theta)}{\sqrt{n\sigma^2(\theta)}} \xrightarrow[n\to\infty]{d} N(0,1), \quad \sigma^2(\theta) = p(\theta)(1-p(\theta))$$

Therefore, when sample size is large, we can approximate the critical region with significance level α as the following :

"$S_n \ge C_n$", $C_n \simeq np(\theta) + \sqrt{n}\,\sigma(\theta_0) z_\alpha$, $p(\theta_0) = 1/2$, $\sigma(\theta_0) = 1/2$

Hence, we can approximate the power of the test for alternative hypothesis $\theta = \theta_1$ $(\theta_1 > \theta_0)$.

$$\begin{aligned}
\gamma_n(\theta_1) &= P_{\theta_1}(S_n \ge C) \\
&= P_{\theta_1}\left(\frac{S_n - np(\theta_1)}{\sqrt{n\sigma^2(\theta_1)}} \ge \frac{C_n - np(\theta_1)}{\sqrt{n\sigma^2(\theta_1)}} \right) \\
&\simeq P_{\theta_1}\left(Z \ge \frac{C_n - np(\theta_1)}{\sqrt{n\sigma^2(\theta_1)}} \right), \quad Z \sim N(0,1) \\
&= 1 - \Phi\left(\frac{C_n - np(\theta_1)}{\sqrt{n\sigma^2(\theta_1)}} \right) \\
&\simeq 1 - \Phi\left\{ -\sqrt{n}\,(p(\theta_1) - p(\theta_0)) / \sigma(\theta_1) + \sigma(\theta_0) z_\alpha / \sigma(\theta_1) \right\}
\end{aligned}$$

Therefore, for a fixed alternative hypothesis $\theta = \theta_1$ $(\theta_1 > \theta_0)$,

$$\lim_{n\to\infty} \gamma_n(\theta_1) = 1$$

and for an alternative hypothesis which approaches to the null hypothesis :

$$\theta_{1n} \simeq \theta_0 + K/\sqrt{n} \quad (K > 0)$$

the following approximation holds:

$$\gamma_n(\theta_{1n}) \simeq 1 - \Phi\left\{ -\sqrt{n}\,(\theta_{1n} - \theta_0)\, p(\theta_0) / \sigma(\theta_0) + z_\alpha \right\}, \quad p(\theta) = \frac{d}{d\theta} P(\theta) = f(\theta_0 - \theta)$$

↳ $\frac{d}{d\theta} P_\theta(X > \theta_0) = \frac{d}{d\theta}\{1 - F(\theta_0 - \theta)\}$
$= f(\theta_0 - \theta)$ ▫

Hence, the sample size needed for $\gamma_{1n}(\theta) \simeq \gamma$ can be asymptotically calculated as the following :

$$\sqrt{n}\,(\theta_{1n} - \theta_0)\, p(\theta_0) / \sigma(\theta_0) - z_\alpha \simeq z_{1-\gamma}$$

$$\therefore n \simeq (2f(0))^{-2} \left(\frac{z_\alpha + z_{1-\gamma}}{\theta_{1n} - \theta_0} \right)^2$$

$$\gamma_n(\theta_{1n}) \simeq 1 - \Phi\left\{ -\sqrt{n}\,(\theta_{1n} - \theta_0)\, p(\theta_0) / \sigma(\theta_0) + z_\alpha \right\}$$

$$\gamma_n(\theta_{1n}) \simeq \Phi\left\{ \sqrt{n}\,(\theta_{1n} - \theta_0)\, p(\theta_0) / \sigma(\theta_0) - z_\alpha \right\}$$

$$\Phi^{-1}(\gamma_n(\theta_{1n})) \simeq \sqrt{n}\,(\theta_{1n} - \theta_0)\, p(\theta_0) / \sigma(\theta_0) - z_\alpha$$

$$* \quad z_{1-\gamma} \simeq \sqrt{n}\,(\theta_{1n} - \theta_0)\, p(\theta_0) / \sigma(\theta_0) - z_\alpha$$

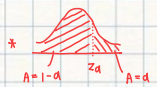
$$\therefore n \simeq \left\{ \frac{z_\alpha + z_{1-\gamma}}{(\theta_{1n} - \theta_0)\, p(\theta_0)} \cdot \sigma(\theta_0) \right\}^2$$

$* \; p(\theta) = f(\theta - \theta_0) \quad \therefore p(\theta_0) = f(\theta_0 - \theta_0) = f(0)$

$* \; \sigma(\theta_0) = \frac{1}{2}$

$\therefore \sigma(\theta_0) \cdot \frac{1}{p(\theta)} = \frac{1}{2f(0)}$

$$\therefore n \simeq (2f(0))^2 \left\{ \frac{z_\alpha + z_{1-\gamma}}{(\theta_{1n} - \theta_0)\, p(\theta_0)} \right\}^2$$



$\therefore \Phi^{-1}(z_\alpha) = 1 - \alpha$

$\Phi^{-1}(z_{1-\alpha}) = \alpha$

* Key takeaways:

1. (condition) Distribution is Continuous & Symmetric (e.g. Normal, Logistic, Double Exponential)
2. Use test statistic $S_n = \sum_{i=1}^{n} I(X_i > \theta_0)$
3. Find Distribution of $S_n \sim Bin(n, p(\theta))$, $p(\theta) = P_\theta(X > \theta_0)$
4. Show that the size of test $\phi(x)$ is an increasing function of θ_0 → so that $\forall\, \theta \le \theta_0$, $E_\theta[\phi(x)] \le \alpha$
5. Use the normal approximation of Binomial Distribution. $(S_n - np(\theta)) / \sqrt{n\sigma^2(\theta)} \sim N(0,1)$, $\sigma^2(\theta_1) = p(\theta)(1 - p(\theta))$
6. Find the critical region C_n s.t. $P_{\theta_0}(S_n \ge C_n) = \alpha$
7. Define the power function $\gamma_n(\theta_1) = P_{\theta_1}[S_n \ge C_n]$
8. For the alternative hypothesis $\theta_{1n} = \theta_0 + K/\sqrt{n}$ $(K > 0)$, find the value of its power
9. Approximate the sample size in order to achieve the desired power.

〈Theorem 9.3.1 Approximation of Power and Sample Size〉

Suppose that the critical region with significance level α for testing the hypothesis:

$$H_0(\theta_0): \theta = \theta_0 \quad vs \quad H_1: \theta > \theta_0$$

is: $\sqrt{n}(T_n - \mu(\theta_0))/\sigma(\theta_0) \geq t_n$

Here, $T_n = T_n(X_1, \cdots, X_n)$, a test statistic, is a function of n random samples.

Also, suppose that T_n is asymptotically normal[132], i.e.,

$$\sqrt{n}(T_n - \mu(\theta))/\sigma(\theta) \xrightarrow[n\to\infty]{d} N(0,1)$$

and $\mu(\theta)$ and $\sigma(\theta)$ are differentiable and continuous.

(a) (Approximation of Power)

For parameter values under the alternative hypothesis that are close to the null hypothesis: i.e.,

$$\theta_{1n} \simeq \theta_0 + k/\sqrt{n} \quad (k>0)$$

the following approximation formula for the power function holds:

$$\gamma_n(\theta_{1n}) = P_{\theta_1}(\sqrt{n}(T_n - \mu(\theta_0))/\sigma(\theta_0) \geq t_n)$$
$$\simeq 1 - \Phi\{-\sqrt{n}(\theta_{1n} - \theta_0)\dot{\mu}(\theta_0)/\sigma(\theta_0) + z_\alpha\}, \quad \dot{\mu}(\theta) = \frac{d}{d\theta}\mu(\theta)$$

(b) (Approximation of Sample Size)

Let $N(T_n; \gamma, \theta_{1n})$ denote the sample size needed for the power of the test under (a), $H_1: \theta = \theta_{1n}$, to approximate to γ: i.e.,

$$\gamma_n(\theta_{1n}) \simeq \gamma$$

The following approximation formula holds:

$$N(T_n; \gamma, \theta_{1n}) \simeq (\dot{\mu}(\theta_0)/\sigma(\theta_0))^{-2}\left(\frac{z_\alpha + z_{1-\gamma}}{\theta_{1n} - \theta_0}\right)^2$$

(proof)

Asymptotic normality holds, and the sample size is a. Thus,

$$P_{\theta_0}\{\sqrt{n}(T_n - \mu(\theta_0))/\sigma(\theta_0) \geq t_n\} = \alpha, \quad P(Z \geq t) = \alpha, \quad Z \sim N(0,1)$$

$$\therefore t_n \simeq z_\alpha$$

Hence, for the power function under the alternative hypothesis: $\theta = \theta_1 (\theta_1 > \theta_0)$, the following approximation holds.

$$\gamma_n(\theta_1) = P_{\theta_1}(\sqrt{n}(T_n - \mu(\theta_0))/\sigma(\theta_0) \geq t_n)$$
$$= P_{\theta_1}\left\{\sqrt{n}\,\frac{T_n - \mu(\theta_1) + \mu(\theta_1) - \mu(\theta_0)}{\sigma(\theta_0)} \geq t_n\right\}$$
$$= P_{\theta_1}\left\{\sqrt{n}\left(\frac{T_n - \mu(\theta_1)}{\sigma(\theta_1)}\cdot\frac{\sigma(\theta_1)}{\sigma(\theta_0)} + \frac{\mu(\theta_1) - \mu(\theta_0)}{\sigma(\theta_0)}\right) \geq t_n\right\}$$
$$= P_{\theta_1}\left\{\sqrt{n}\,\frac{T_n - \mu(\theta_1)}{\sigma(\theta_1)}\cdot\frac{\sigma(\theta_1)}{\sigma(\theta_0)} \geq z_n - \sqrt{n}\,\frac{\mu(\theta_1) - \mu(\theta_0)}{\sigma(\theta_0)}\right\}$$
$$= P_{\theta_1}\left\{\sqrt{n}\,\frac{T_n - \mu(\theta_1)}{\sigma(\theta_1)} \geq \frac{\sigma(\theta_0)}{\sigma(\theta_1)}z_n - \sqrt{n}\,\frac{\mu(\theta_1) - \mu(\theta_0)}{\sigma(\theta_1)}\right\}$$
$$= P_{\theta_1}\left\{Z > \frac{\sigma(\theta_0)}{\sigma(\theta_1)}z_\alpha - \sqrt{n}\,\frac{\mu(\theta_1) - \mu(\theta_0)}{\sigma(\theta_1)}\right\}, \quad Z \sim N(0,1)$$
$$= 1 - \Phi\left\{\frac{\sigma(\theta_0)}{\sigma(\theta_1)}z_\alpha - \sqrt{n}\,\frac{\mu(\theta_1) - \mu(\theta_0)}{\sigma(\theta_1)}\right\}$$

Hence, we can approximate the power of the test under $\theta_{1n} \simeq \theta_0 + k/\sqrt{n}\ (k>0)$ as the following:

$$\gamma_n(\theta_{1n}) \simeq 1 - \Phi\{-\sqrt{n}(\theta_{1n} - \theta_0)\dot{\mu}(\theta_0)/\sigma(\theta_0) + z_\alpha\}$$

* Since $\mu(\theta)$ is differentiable,

[Taylor series expansion]

$\mu(\theta_{1n}) = \mu(\theta_0) + \dot{\mu}(\theta_0)(\theta_{1n} - \theta_0) + o(n^{-1/2})$, $\sqrt{n}\,o(n^{-1/2}) \xrightarrow[n\to\infty]{P_\theta} 0$

$$\therefore \gamma_n(\theta_{1n}) = 1 - \Phi\left\{\frac{\sigma(\theta_0)}{\sigma(\theta_1)}z_\alpha - \sqrt{n}\,\frac{\mu(\theta_0) + \dot{\mu}(\theta_0)(\theta_{1n} - \theta_0) + o(n^{-1/2}) - \mu(\theta_0)}{\sigma(\theta_1)}\right\}$$
$$= 1 - \Phi\left\{\frac{\sigma(\theta_0)}{\sigma(\theta_1)}z_\alpha - \sqrt{n}\,\frac{\dot{\mu}(\theta_0)(\theta_{1n} - \theta_0) + o(n^{-1/2})}{\sigma(\theta_1)}\right\}$$
$$\simeq 1 - \Phi\left\{z_\alpha - \sqrt{n}\,\frac{\dot{\mu}(\theta_0)(\theta_{1n} - \theta_0)}{\sigma(\theta_1)}\right\} \quad \because \sqrt{n}\,o(n^{-1/2}) \to 0 \text{ and } \frac{\sigma(\theta_0)}{\sigma(\theta_{1n})} \to 1$$

Also we can get the approximation formula of sample size $N(T_n; \gamma, \theta_{1n})$ from the following approximation calculation (근사계산).

$$\sqrt{N(T_n; \gamma, \theta_{1n})}\,(\theta_{1n} - \theta_0)\dot{\mu}(\theta_0)/\sigma(\theta_0) - z_\alpha \simeq z_{1-\gamma} \quad *$$

* n을 $N(T_n; \gamma, \theta_{1n})$으로 교환 후 정리.

*

$$1 - \Phi\left\{z_\alpha - \sqrt{n}\,\frac{\dot{\mu}(\theta_0)(\theta_{1n} - \theta_0)}{\sigma(\theta_1)}\right\} \simeq \gamma$$

$$\Leftrightarrow \Phi\left\{z_\alpha - \sqrt{n}\,\frac{\dot{\mu}(\theta_0)(\theta_{1n} - \theta_0)}{\sigma(\theta_1)}\right) \simeq 1 - \gamma$$

$$\Leftrightarrow z_\alpha - \sqrt{n}\,\frac{\dot{\mu}(\theta_0)(\theta_{1n} - \theta_0)}{\sigma(\theta_1)} \simeq z_\gamma, \quad \Phi(z_\gamma) = 1 - \gamma$$

$$\Leftrightarrow -z_\alpha + \sqrt{n}\,\frac{\dot{\mu}(\theta_0)(\theta_{1n} - \theta_0)}{\sigma(\theta_1)} \simeq z_{1-\gamma}$$

Definition: Asymptotic Relative Efficiency.

Let $T_{in}\ (i=1,2)$ denote the test statistics, composed of n random samples, for testing the hypothesis:

$$H_0(\theta_0): \theta = \theta_0 \quad vs \quad H_1: \theta > \theta_1$$

at significance level $\alpha\ (0<\alpha<1)$. $T_{in}\ (i=1,2)$ satisfies the conditions in ⟨Theorem 9.3.1⟩.

Let $N(T_{in}; \gamma, \theta_{1n})\ (i=1,2)$ denote the sample size needed for the power of the test under $H_1: \theta = \theta_{1n}\ (\theta_{1n} \simeq +K/\sqrt{n},\ K>0)$ to achieve a power γ. Then

$$\lim_{n\to\infty} N^{-1}(T_{1n}; \gamma, \theta_{1n}) ; \lim_{n\to\infty} N^{-1}(T_{2n}; \gamma, \theta_{1n})$$

is called the ARE (Asymptotic Relative Efficiency) of 'Test method 1 based on T_{1n}' w.r.t. 'Test method 2 based on T_{2n}'. Such ARE is denoted as: $ARE(\{T_{1n}\}, \{T_{2n}\})$ and from ⟨Theorem 9.3.1⟩, it is expressed as:

$$ARE(\{T_{1n}\}, \{T_{2n}\}) = \lim_{n\to\infty} N^{-1}(T_{1n}; \gamma, \theta_{1n}) / N^{-1}(T_{2n}; \gamma, \theta_{1n})$$

$$= \left(\mu_1'(\theta_0)/\sigma_1(\theta_0)\right)^2 / \left(\mu_2'(\theta_0)/\sigma_2(\theta_0)\right)^2$$

Ex. 9.3.3. ARE of the Sign test relative to the t-test.

In the case where the population distribution is continuous, symmetric about μ and its pdf is:

$$\frac{1}{\sigma} f\left(\frac{x-\mu}{\sigma}\right),\ -\infty<\mu<+\infty,\ \sigma>0$$

Consider testing the hypothesis:

$$H_0(\mu_0): \mu = \mu_0 \quad vs \quad H_1: \mu > \mu_0$$

at significance level $\alpha\ (0<\alpha<1)$ using the test statistic

$$S_n = \sum_{i=1}^{n} I(X_i > \mu_0)$$

The sign test using S_n is similar to that of ⟨Ex. 9.3.1⟩

From ⟨Ex. 9.3.2⟩, we know that the required sample size for power of the test under $H_1: \mu = \mu_1\ (\simeq \mu_0 + K/\sqrt{n},\ K>0)$ to achieve power γ is:

$$N(S_n; \gamma, \mu_{1n}) \simeq \underbrace{(2f(0))^2}_{(*)} \sigma^{-2} \left(\frac{Z_\alpha + Z_{1-\gamma}}{\mu_{1n} - \mu_0}\right)^2$$

(*) $T_n = \frac{1}{n}\sum_{i=1}^{n} I(X_i > \mu_0)$, $\mu(\theta) = E_\theta[I(X_i > \mu_0)]$, $\sigma(\theta)^2 = Var_\theta[I(X_i > \mu_0)]$

$$\mu'(\theta_0) = \left[\frac{\partial}{\partial\theta}\mu(\theta)\right]_{\theta=\theta_0} = \left[\frac{\partial}{\partial\theta} P(X>\mu_0)\right]_{\theta=\theta_0} = \left[\frac{\partial}{\partial\theta}\left(1 - P\left(\frac{X-\mu}{\sigma} < \frac{\mu_0-\mu}{\sigma}\right)\right)\right]_{\theta=\theta_0}$$

$$= \left[\frac{\partial}{\partial\theta}\left(1 - F\left(\frac{\mu_0-\mu}{\sigma}\right)\right)\right]_{\theta=\theta_0}$$

$$= \left[\frac{1}{\sigma} f\left(\mu_0-\mu\right)\right]_{\theta=\theta_0} = \sigma^{-1} f(0)$$

* $\theta = \mu$ & $\theta_0 = \mu_0$

$$\sigma(\theta)^2 = p(\theta_0)(1-p(\theta_0)) = P(\bar{X}>\theta_0)[1-P(\bar{X}>\theta_0)]$$
$$= \left[1-P_{\theta_0}\left(\frac{\bar{X}-\mu}{\sigma} < \frac{\mu_0+\mu}{\sigma}\right)\right] P_{\theta_0}\left(\frac{\bar{X}-\mu}{\sigma} < \frac{\mu_0-\mu}{\sigma}\right)$$
$$= \left\{1-F\left(\frac{(\mu_0-\mu)}{\sigma}\right)\right\}\left\{F\left(\frac{\mu_0-\mu}{\sigma}\right)\right\}$$

$\sigma(\theta)^2\big|_{\theta=\theta_0} = (1-F(0))F(0) = \frac{1}{2}\cdot\frac{1}{2} = 1/4$

$\therefore \sigma(\theta_0) = 1/2$ $\quad\Rightarrow \dot{\mu}(\theta_0)/\sigma(\theta_0) = \left[f(0)/\sigma\right]/\frac{1}{2} = 2f(0)/\sigma$

Meanwhile, if the population distribution is normal, we would typically use the t-test given as:

"$\sqrt{n}\,|\bar{x}-\mu_0|/s \geq t_\alpha(n-1)$"

For such t-test statistic, $T_n = \sqrt{n}\,|\bar{x}-\mu_0|/S$, we know that the following asymptotic normality holds:[133]

$$\sqrt{n}\left(\frac{\bar{X}-\mu_0}{S} - \frac{\mu_{in}-\mu_0}{\sqrt{Var(X_i)}}\right) \xrightarrow[n\to\infty]{d} N(0,1), \quad \sqrt{Var(X)} = \sigma\left(\int_{-\infty}^{+\infty} z^2 f(z)\,dz\right)^{1/2}$$

under H_1 $\sqrt{n}\left(\frac{\bar{X}-\mu_1}{\sigma(\mu_1)}\right) \longrightarrow N(0,1)$

$$\therefore \sqrt{n}\left(\frac{\bar{X}-\mu_0+\mu_0-\mu_1}{\sigma(\mu_1)}\right) = \sqrt{n}\left(\frac{\bar{X}+\mu_0}{\sigma(\mu_1)} - \frac{\mu_1-\mu_0}{\sigma(\mu_1)}\right)$$
$$= \sqrt{n}\left(\frac{\bar{X}-\mu_0}{S}\cdot\frac{S}{\sigma(\mu_1)} - \frac{\mu_1-\mu_0}{\sigma(\mu_1)}\right)$$
$$= \sqrt{n}\left(\frac{\bar{X}-\mu_0}{S} - \frac{\mu_1-\mu_0}{\sqrt{Var(X)}}\right) \longrightarrow N(0,1)$$

Therefore, from <Theorem 9.3.1> the approximation of sample size in t-test is:

$$N(T_n; \gamma, \mu_{in}) \simeq \left(1/\sqrt{Var(X)}\right)^{-2}\left(\frac{z_{\alpha+z_{1-\gamma}}}{\mu_{in}-\mu_0}\right)^2$$

Hence the ARE of sign test relative to the t-test is:

$$ARE(\{S_n\},\{T_n\}) = (2f(0)/\sigma)^2\left(1/\sqrt{Var(X_i)}\right)^{-2} = 4(f(0))^2\int_{-\infty}^{+\infty} z^2 f(z)\,dz$$

and we see that it is equal to the ARE of sample median w.r.t sample mean from <Ex. 8.4.3>.

Table 9.3.1. ARE of Sign test w.r.t. t-test.

population Distribution	$N(\mu,\sigma^2)$	$L(\mu,\sigma)$	$DE(\mu,\sigma)$
$ARE(\{S_n\},\{T_n\})$	$2/\pi = 0.636$	$\pi^2/12 = 0.822\cdots$	2

Testing procedures that can be applied without assuming a specific functional form for the population distribution, such as in the sign test, are called nonparametric tests. These methods are useful when it is difficult to assess the validity of assumptions about the population distribution. Let us examine methods based on the use of rank order, which are among the most representative nonparametric testing procedures.

133) 모평균이 $\mu_{in} \simeq \mu_0 + k/\sqrt{n}$ $(k>0)$ 일 때의 극한분포를 뜻하는 것이다.

Ex. 9.3.4. Signed rank test statistic in location parameter model.

Similar to <Ex. 9.3.1>, consider a model whose population distribution is continuous, its pdf is:

$$f(x-\theta), \quad -\infty < \theta < +\infty$$

and is symmetric about θ, i.e. $f(-x) = f(x) \;\forall x: -\infty < x < +\infty$

Consider testing the hypothesis:

$$H_0(\theta_0): \theta = \theta_0 \quad vs \quad H_1: \theta > \theta_0$$

at significance level $\alpha\,(0<\alpha<1)$ using the random samples $X_1, \cdots, X_n$

The rank of $|X_i-\theta|$ refers to the order of the size of $|X_i-\theta|$ when $|X_1-\theta_0|, \cdots, |X_n-\theta_0|$ are arranged in an increasing order. The rank of $|X_i-\theta|$ is denoted as $R(|X_i-\theta_0|)$ and

$$R(|X_i-\theta_0|) = 1 + \sum_{j=1}^{n} I(|X_j-\theta_0| < |X_i-\theta_0|)$$

The following statistic is called the 'Signed rank' test statistic

$$W_n = \sum_{i=1}^{n} sgn(X_i-\theta)R(|X_i-\theta_0|)$$

Here, $sgn(x)$ is a function that outputs the sign(부호) of x.

☆ Theorem 9.3.2. The Distribution of Signed rank test statistic under H_0

The following holds for the signed rank test statistic's distribution under the null hypothesis, $H_0(\theta_0): \theta = \theta_0$

(a) $W_n \overset{d}{=} \sum_{j=1}^{n} jS(j)$, $S(j) \overset{iid}{\sim}$, $P(S(j)=-1) = P(S(j)=+1) = 1/2$ $\quad (j=1,2,\cdots)$

(b) $(W_n - E_{\theta_0}W_n)/\sqrt{Var_{\theta_0}(W_n)} \xrightarrow[n\to\infty]{d} N(0,1)$

$$E_{\theta_0}[W_n] = 0, \quad Var_{\theta_0}(W_n) = \sum_{i=1}^{n} j^2 = n(n+1)(2n+1)/6$$

(proof)

We will denote the sign and rank vectors as the following:

$$S = (S(1), \cdots, S(n))^t = (sgn(X_1-\theta_1), \cdots, sgn(X_n-\theta_0))^t,$$

$$R = (R(1), \cdots, R(n))^t = (R|X_1-\theta_0|), \cdots, R(|X_n-\theta_0|)^t$$

(a) First, under the null hypothesis $H_0(\theta_0): \theta = \theta_0$, the distribution of X is symmetric about θ_0. Hence,

$$P_{\theta_0}(|X_i-\theta_0| \leq x, \; sgn(X_i-\theta_0) = +1)$$
$$= P_{\theta_0}(0 < X_i-\theta_0 \leq x)$$
$$= \tfrac{1}{2}P_{\theta_0}(|X_i-\theta_0| \leq x)$$
$$= P_{\theta_0}(|X_i-\theta_0| \leq x)\,P_{\theta_0}(sgn(X_i-\theta_0) = +1),$$
$$P_{\theta_0}(|X_i-\theta_0| \leq x, \; sgn(X_i-\theta_0) = -1)$$
$$= P_{\theta_0}(|X_i-\theta_0| \leq x)\,P_{\theta_0}(sgn(X_i-\theta_0) = -1)$$

Meaning, under the null hypothesis $H_0|\theta_0): \theta=\theta_0$, $|X_i-\theta_0|$ and $\text{sgn}(X_i-\theta_0)$ are mutually independent, and the sign vector $S=(S(1),\cdots,S(n))^t$ and rank vector $R=(R(1),\cdots,R(n))^t$ are mutually independent.

Secondly, if $R(i)=j$, let $R^{-1}(j)=i$ and let $R^{-1}=(R^{-1}(1),\cdots,R^{-1}(n))$ be the inverse rank vector (역순위 벡터). We can express the signed rank test statistic as the following:

$$W_n=\sum_{i=1}^n \text{sgn}(X_i-\theta_0)R(|X_i-\theta_0|)=\sum_{i=1}^n S(i)R(i)=\sum_{j=1}^n S(R^{-1}(j))\,j$$

Meanwhile since $X_1,\cdots,X_n$ are iid, the following holds for an arbitrary substitution π for $\{1,2,\cdots,n\}$

$$(S(\pi^{-1}(j)))_{1\le j\le n}=(\text{sgn}(X_{\pi^{-1}(j)}-\theta_0))_{1\le j\le n}\overset{d}{\equiv}(\text{Sgn}(X_j-\theta_0))_{1\le j\le n}$$

$$=(S(j))_{1\le j\le n}$$

Also, under the null hypothesis, from the independence of $S=(S(1),\cdots,S(n))^t$ and $R=(R(1),\cdots,R(n))^t$, the following holds:

$$P_{\theta_0}(S(R^{-1}(1))=s_1,\cdots,S(R^{-1}(n))=s_n)$$

$$=\sum_{\pi\in\Pi}P_{\theta_0}(S(R^{-1}(1))=s_1,\cdots,S(R^{-1}(n))=s_n,\ R=\pi)$$

$$=\sum_{\pi\in\Pi}P_{\theta_0}(S(\pi^{-1}(1))=s_1,\cdots,S(\pi^{-1}(n))=s_n)\,P_{\theta_0}(R=\pi)$$

$$=\sum_{\pi\in\Pi}P_{\theta_0}(S(1)=s_1,\cdots,S(n)=s_n)\,P_{\theta_0}(R=\pi)$$

$$=P_{\theta_0}(S(1)=s_1,\cdots,S(n)=s_n)$$

* Revision needed → alternative approach

Therefore, under the null hypothesis $H_0|\theta_0): \theta=\theta_0$, the following holds:

$$W_n=\sum_{j=1}^n S(R^{-1}(j))\,j\overset{d}{\equiv}\sum_{j=1}^n j\cdot S(j)$$

Also, under the null hypothesis $H_0|\theta_0): \theta=\theta_0$, because the distribution of X_j is symmetric about θ_0,

$$P_{\theta_0}(S(j)=-1)=P_{\theta_0}(S(j)=+1)=1/2$$

and (a) is proved.

(b) Let $\sigma_n=\sqrt{\text{Var}_{\theta_0}(W_n)}$, $Z_n=W_n/\sigma_n$. From (a),

$$E_{\theta_0}[W_n]=0,\quad \sigma_n^2=\text{Var}_{\theta_0}(W_n)=\sum_{j=1}^n j^2=n(n+1)(2n+1)/6$$

Also, from (a), under the null hypothesis $H_0|\theta_0): \theta=\theta_0$, we can approximate the cgf of Z_n as the following:

$$\text{cgf}_{Z_n}(t)=\sum_{j=1}^n \log\{(\exp(-jt/\sigma_n)+\exp(jt/\sigma_n))/2\}$$

$$=\sum_{j=1}^n \log\left\{1+\frac{1}{2}\frac{j^2}{\sigma_n^2}t^2+\frac{1}{4!}\frac{j^4}{\sigma_n^4}t^4+\cdots\right\}$$

$$=\sum_{j=1}^n\left\{\left(\frac{1}{2}\frac{j^2}{\sigma_n^2}t^2+\frac{1}{4!}\frac{j^4}{\sigma_n^4}t^4+\cdots\right)-\frac{1}{2}\left(\frac{1}{2}\frac{j^2}{\sigma_n^2}t^2+\cdots\right)^2+\cdots\right\}$$

$$\simeq\sum_{j=1}^n\frac{1}{2}\frac{j^2}{\sigma_n^2}t^2+\cdots\simeq\frac{1}{2}t^2+\cdots$$

Hence, under the null hypothesis $H_0|\theta_0): \theta=\theta_0$, the asymptotic normality of W_n holds. i.e.,

$$(W_n-E_{\theta_0}[W_n])/\sqrt{\text{Var}_{\theta_0}(W_n)}=Z_n\xrightarrow[n\to\infty]{d}N(0,1)$$

(My proof)

$$P_{\theta_0}(|X_i-\theta_0|\le x,\ \text{sgn}(X_i-\theta_0)=+1)$$
$$=P_{\theta_0}(-x\le X_i-\theta_0\le x,\ X_i-\theta_0>0)$$
$$=P_{\theta_0}(0<X_i-\theta_0\le x)$$
$$=\frac{1}{2}P_{\theta_0}(-x\le X_i-\theta_0\le x)$$
$$=\frac{1}{2}P_{\theta_0}(|X_i-\theta_0|\le x)$$
$$=P_{\theta_0}(|X_i-\theta_0|\le x)\,P_{\theta_0}(\text{sgn}(X_i-\theta_0)=+1)\quad\text{(or)}$$

① Show independence

Similarly, $P_{\theta_0}(|X_i-\theta_0|\le x,\text{sgn}(X_i-\theta_0)=-1)$
$=P_{\theta_0}(|X_i-\theta_0|\le x)\,P_{\theta_0}(\text{Sgn}(X_i-\theta_0)=-1)$

$\therefore$ under $H_0|\theta_0): \theta=\theta_0$, $|X_i-\theta_0|$ and $\text{sgn}(X_i-\theta_0)$ are independent

$\Rightarrow R=(R(1),\cdots,R(n))^t$ and $S=(S(1),\cdots,S(n))^t$ are independent because R is a function of $|X_i-\theta|$ $(i=1,\cdots,n)$

$$W_n=\sum_{j=1}^n \text{sgn}(X_i-\theta_0)R(|X_i-\theta|)=\sum_{i=1}^n S(i)R(i)=\sum_{j=1}^n S(R^{-1}(j))\,j$$

② Express using vectors

③ use inverse permutation $R^{-1}(\cdot)$

$$=\sum_{j=1}^n S(j)\cdot j$$

④ Show that $R^{-1}(j)=j$

[produced by ChatGPT 5.0]

* $S(j)$: Sign of the observation whose rank is j

Which means $S(j)=\text{sgn}(X_{i_j}-\theta_0)$ where i_j is the index satisfying $R(i_j)=j$

By the definition of inverse permutation, $R^{-1}(j)=i_j$

Therefore, $S(j)=\text{sgn}(X_{R^{-1}(j)}-\theta_0)=S(R^{-1}(j))$

$\therefore S(R^{-1}(j))=S(j)$

$\therefore W_n\overset{d}{\equiv}\sum_{j=1}^n S(j)\cdot j$

Also under $H_0|\theta_0): \theta=\theta_0$, the distribution of X_j is symmetric about θ_0. Thus,

$$P_{\theta_0}(S(j)=-1)=P_{\theta_0}(S(j)=+1)=1/2$$

(b) $\text{Var}(W_n)=\text{Var}(\sum_{j=1}^n j\,S(j))=\text{Var}(\sum_{j=1}^n j)=\sum_{i=1}^n\text{Var}(j)=\sum_{i=1}^n j^2\,\text{Var}(1)$

$$=\sum_{j=1}^n j^2$$

$$E[W_n]=E[\sum_{j=1}^n j\cdot S(j)]=\frac{1}{2}E[\sum_{j=1}^n j]+\frac{1}{2}E[\sum_{j=1}^n -j]=0$$

$$M_{W_n}(t) = E[e^{tW_n}] = E\left[\exp\left\{t\sum_{j=1}^{n} j\,S(j)\right\}\right]$$

$$C_{W_n}(t) = \log(E[e^{tW_n}])$$

$$= \log\left(E\left[\exp\left\{t\sum_{j=1}^{n} j\,S(j)\right\}\right]\right)$$

$$= \sum_{j=1}^{n} \log\left(E[\exp\{t\,j\,S(j)\}]\right)$$

$$= \sum_{j=1}^{n} \log\left\{\frac{1}{2}(\exp\{tj\} + \exp\{-tj\})\right\}$$ ← Taylor series expansion

$$= \sum_{j=1}^{n} \log\left\{\frac{1}{2}\left(1 + jt + \frac{j^2}{2!}t^2 + \cdots + 1 - jt + \frac{j^2}{2!} + \cdots\right)\right\}$$

$$= \sum_{j=1}^{n} \log\left(1 + \frac{j^2}{2!}t^2 + \cdots\right)$$

$$\simeq \sum_{j=1}^{n}\left(\frac{j^2}{2!}t^2 + \cdots\right) - \frac{1}{2}\left(\frac{j^2}{2!}t + \cdots\right)^2 + \cdots \quad (\because \log(1+A) = A - \frac{1}{2}A^2 + \cdots$$

$$\simeq \sum_{j=1}^{n} \frac{j^2}{2}t^2 = \frac{t^2}{2}\sum_{j=1}^{n} j^2$$

$$C_{W_n/\sigma}(t) = C_{W_n}(t/\sigma) = \frac{t^2}{2\sigma^2}\sum_{j=1}^{n} j^2 = \frac{t^2}{2}\left(\because \sigma^2 = Var(W_n) = \sum_{j=1}^{n} j^2\right)$$

cgf of $N(0,1) = t^2/2 \quad \therefore \frac{W_n}{\sigma} \sim N(0,1) \Leftrightarrow \frac{W_n - E[W_n]}{\sigma} \sim N(0,1)$ □

〈Theorem 9.3.3. Expression of signed rank test statistic〉

Let $W_n^+ = \sum_{i=1}^{n} I(X_i - \theta_0 > 0)\,R(|X_i - \theta_0|)$

Then the following holds:

(a) $W_n = 2W_n^+ - n(n+1)/2$

(b) under the null hypothesis $H_0(\theta_0): \theta = \theta_0$,

$$W_n^+ \overset{d}{\equiv} \sum_{j=1}^{n} j\,B_j, \quad B_j \overset{iid}{\sim} \text{Bernoulli}(1/2), \quad (j = 1, \cdots, n)$$

$$(W_n^+ - n(n+1)/4)/\sqrt{n(n+1)(2n+1)/24} \xrightarrow[n\to\infty]{d} N(0,1)$$

$$\Rightarrow \{W_n\}/\sqrt{n(n+1)(2n+1)/6} \xrightarrow[n\to\infty]{d} N(0,1)$$

(proof) $\Rightarrow \{2W_n^+ - n(n+1)/2\}/\sqrt{n(n+1)(2n+1)/6} \xrightarrow[n\to\infty]{d} N(0,1)$ ← divide by 2 on denominator & numerator

Since the population distribution is continuous,

$$\text{sgn}(X_i - \theta_0) = 2I(X_i - \theta_0 > 0) - 1$$

Hence, inserting the above to $W_n = \sum_{i=1}^{n} \text{sgn}(X_i - \theta_0)\,R(|X_i - \theta_0|)$ will give us (a). Also from 〈Theorem 9.3.2〉-(a)

$\{S(j)+1\}/2 \sim \text{Bernoulli}(1/2)$

$$W_n^+ = (W_n + n(n+1)/2)/2 \overset{d}{\equiv} \sum_{j=1}^{n} j\,(S(j)+1)/2$$ *

$S(j) = \{-1, 1\}$

$S(j)+1 = \{0, 2\}$

$\therefore (S(j)+1)/2 = \{0, 1\}$

From 〈Theorem 9.3.2〉, under $H_0(\theta_0): \theta = \theta_0$, the signed rank test statistic W_n's distribution is independent of the pdf of the population distribution. and large value of W_n can be considered as an evidence for the alternative hypothesis $H_1: \theta > \theta_0$. Therefore, the signed-rank test with significance level α for testing: ↳ large $W_n \Rightarrow$ many $X_i - \theta_0 > 0 \Rightarrow X_i > \theta_0$

$$H_0(\theta_0): \theta = \theta_0 \quad \text{vs} \quad H_1: \theta > \theta_1$$

is,

$$\phi_{SR}(X_1, \cdots, X_n) = \begin{cases} 1, & W_n \geq C+1 \\ \gamma, & W_n = C \\ 0, & W_n \leq C-1 \end{cases} \quad (0 \leq \gamma \leq 1), \quad E_{\theta_0}[\phi_{SR}(X)] = \alpha$$

Also, from 〈Theorem 9.3.3〉, we can express the above using W_n^+:

$$\phi_{SR}(X_1, \cdots, X_n) = \begin{cases} 1, & W_n^+ \geq C^+ - 1 \\ \gamma, & W_n^+ = C^+ \\ 0, & W_n^+ \leq C^+ - 1 \end{cases} \quad (0 \leq \gamma \leq 1), \quad E_{\theta_0}[\phi_{SR}(X)] = \alpha$$

The cumulative probabilities of W_n or W_n^+ can be calculated using R, introduced in Appendix Ⅲ.

Meanwhile, from 〈Theorem 9.3.2〉 and 〈Theorem 9.3.3〉, the critical region with significance level α can be approximated as the following when the sample size is large.

"$W_n/\sqrt{n(n+1)(2n+1)/6} \geq z_\alpha$"

or "$(W_n^+ - n(n+1)/4)/\sqrt{n(n+1)(2n+1)/24} \geq z_\alpha$"

$W_n = 2W_n^+ - \frac{n(n+1)}{2}$

〈Theorem 9.3.4. Signed rank test for one-sided hypothesis〉

Consider a model whose population distribution is continuous, its pdf is :

$$f(x-\theta),\ -\infty<\theta<+\infty$$

and is symmetric about θ, i.e. $f(-x)=f(x)\ \forall x: -\infty<x<+\infty$

Consider testing the hypothesis:

$$H_0(\theta_0): \theta=\theta_0 \quad vs \quad H_1: \theta>\theta_0$$

at significance level $\alpha\ (0<\alpha<1)$ using the random samples $X_1,\cdots,X_n$
The following holds for the signed rank test ϕ_{SR}

(a) $W_n^+ = \sum\sum_{1\le i\le j\le n} I(X_i+X_j>2\theta_0)$

(b) The power function $\gamma_{\phi_{SR}}(\theta)=E_\theta\phi_{SR}(X)$ is an increasing function of θ and the following holds:

$$\max_{\theta\le\theta_0} E_\theta[\phi_{SR}(X)] = E_{\theta_0}[\phi_{SR}(X)] = \alpha$$

(proof)

(a) Using $Y_i=X_i-\theta_0$, we can express W^+ as the following:

$$\begin{aligned}
W_n^+ &= \sum_{i=1}^n I(Y_i>0)R(|Y_i|) \\
&= \sum_{i=1}^n I(Y_i>0)\Big\{1+\sum_{j=1}^n I(|Y_j|<|Y_i|)\Big\} \\
&= \sum_{i=1}^n I(Y_i>0) + \sum_{i=1}^n\sum_{j=1}^n I(Y_i>0, -Y_i<Y_j<Y_i) \\
&= \sum_{i=1}^n I(Y_{(i)}>0) + \sum_{i=1}^n\sum_{j<i}^n I(Y_{(i)}>0, -Y_{(i)}<Y_{(j)}<Y_{(i)}) \\
&= \sum_{i=1}^n I(Y_{(i)}>0) + \sum_{i=1}^n\sum_{j<i}^n I(Y_{(j)}+Y_{(i)}>0) \\
&= \sum_{i=1}^n I(Y_i>0) + \sum_{i=1}^n\sum_{j<i} I(Y_j+Y_i>0) \\
&= \sum\sum_{1\le j<i\le n} I(Y_i+Y_j>0) = \sum\sum_{1\le j<i\le n} I(X_i-\theta_0+X_j-\theta_0>0) = \sum\sum_{1\le j<i\le n} I(X_i+X_j>2\theta_0) \\
&= \sum\sum_{1\le i\le j\le n} I(Y_i+Y_j>0) \quad (\because\ i\leftrightarrow j \text{ doesn't change the equation})
\end{aligned}$$

(b) From the definition of signed rank test :

$$\phi_{SR}(X_1,\cdots,X_n) = \begin{cases} 1, & W_n^+ \ge c^+ +1 \\ \gamma, & W_n^+ = c^+ \quad (0\le\gamma\le 1) \\ 0, & W_n^+ \le c^+ -1 \end{cases}$$

$\phi_{SR}(X_1,\cdots,X_n)$ is an increasing function of W_n^+. However, from (a),

$$W_n^+ = \sum\sum_{1\le i\le j\le n} I(X_i+X_j>2\theta_0)$$

Thus, W_n^+ is an increasing function for each $X_i\ (i=1,2,\cdots,n)$. Hence, $\phi_{SR}(X_1,\cdots,X_n)$ is an increasing function for each $X_i\ (i=1,2,\cdots,n)$. Meanwhile, the power function of the test is:

X_i increases $\Rightarrow$ W_n^+ increases $\Rightarrow$ $E_\theta[\phi_{SR}(X_1,\cdots,X_n)]$ increases

$\because E_\theta[\phi_{SR}(X_1,\cdots,X_n)] = 1\cdot P(W_n^+\ge c^+ +1) + \gamma P(W_n^+ = c^+)$

$$\gamma_{\phi_{SR}}(\theta) = E_\theta[\phi_{SR}(X_1,\cdots,X_n)] = E\,\phi_{SR}(Z_1+\theta,\cdots,Z_n+\theta),\ Z_i \overset{iid}{\sim} f\ (i=1,2,\cdots,n)$$

Thus, the power function $\gamma_{\phi_{SR}}(\theta)$ is an increasing function of θ and (b) holds.

As can be seen from 〈Theorem 9.3.4〉, the signed rank test can be used for testing both one-sided and two-sided hypotheses concerning the median of a continuous symmetric distribution. This is a nonparametric test that can be applied without assuming a specific form of the population distribution. To examine the efficiency of this signed-rank test, the following asymptotic normality is required. The proof of this theorem is beyond the scope of this book and is therefore omitted.

(My proof)

$* R(|X_i-\theta_0|) = 1+\sum_{j=1}^n I(|X_j-\theta_0|<|X_i-\theta_0|)$

(a) $W_n^+ = \sum_{i=1}^n I(Y_i>0)R(|Y_i|)$

$$\begin{aligned}
&= \sum_{i=1}^n I(Y_i>0)\Big\{1+\sum_{j=1}^n I(|Y_j|<|Y_i|)\Big\} \\
&= \sum_{i=1}^n I(Y_i>0) + \sum_{i=1}^n\sum_{j=1}^n I(Y_i>0)I(-Y_i<Y_j<Y_i) \\
&= \sum_{i=1}^n I(Y_i>0) + \sum_{i=1}^n\sum_{j=1}^n I(Y_i>0, -Y_i<Y_j<Y_i) \\
&= \sum_{i=1}^n I(Y_{(i)}>0) + \sum_{i=1}^n\sum_{j=1}^n I(Y_{(i)}>0, -Y_{(i)}<Y_{(j)}<Y_{(i)}) \\
&= \sum_{i=1}^n I(Y_{(1)}>0) + \sum_{i=1}^n\sum_{j<i}^n I(Y_{(1)}>0, -Y_{(i)}<Y_{(j)}) \\
&= \sum_{i=1}^n I(Y_{(i)}>0) + \sum_{i=1}^n\sum_{j<i}^n I(Y_{(i)}+Y_{(j)}>0) \\
&= \sum_{i=1}^n\sum_{j=i}^n I(Y_{(i)}+Y_{(j)}>0) + \sum_{i=1}^n\sum_{j<i}^n I(Y_{(i)}+Y_{(j)}>0) \\
&= \sum_{i=1}^n\sum_{j\le i}^n I(Y_{(i)}+Y_{(j)}>0) \\
&= \sum\sum_{1\le j\le i\le n} I(Y_{(i)}+Y_{(j)}>0) \\
&= \sum\sum_{1\le j\le i\le n} I(Y_i+Y_j>0) = \sum\sum_{1\le i\le j\le n} I(Y_i+Y_j>0)
\end{aligned}$$

(b)

From (a), we know that

$$W_n^+ = \sum\sum_{1\le j\le i\le n} I(Y_i+Y_j>0) = \sum\sum_{1\le j\le i\le n} I(X_i-\theta_0+X_j-\theta_0>0)$$

$$= \sum\sum_{1\le j\le i\le n} I(X_i+X_j>2\theta_0)$$

$\therefore$ W_n^+ is an increasing function of $X_i\ (i=1,\cdots,n)$

$\Rightarrow$ $\phi_{SR}(X_1,\cdots,X_n)$ is an increasing function of W_n^+

$\Rightarrow$ $\therefore$ $\phi_{SR}(X_1,\cdots,X_n)$ is an increasing function of $X_i\ (i=1,\cdots,n)$

☆ 〈강의 9.3.5. Asymptotic Normality of Signed-rank test〉

In a Signed-rank test, the following holds for the test statistic W_n^+.

(a) $\dfrac{W_n^+ - E_\theta(W_n^+)}{\sqrt{Var_\theta(W_n^+)}} \xrightarrow[n\to\infty]{d} N(0,1)$

(b) $\mu(\theta) = \frac{1}{2} P_\theta(X_1+X_2 > 2\theta_0)$

Let $\sigma^2(\theta) = Cov_\theta(I(X_1+X_2>2\theta_0), I(X_1+X_3>2\theta_0))$,

then $E_\theta(W_n^+) \simeq n^2\mu(\theta)$, $Var_\theta(W_n^+) \simeq n^3\sigma^2(\theta)$

} Accept it as it is.

From 〈Theorem 9.3.5〉 we can apply 〈Theorem 9.3.1〉 to the statistic W_n^+/n^2. Thus, we can get the approximation formula for the power of the Signed-rank test as the following:

why use W_n^+/n^2 : I'm guessing it has something to do with eliminating the n terms in $E[W_n^+] = n(n+1)/4$ and $Var(W_n^+) = n(n+1)(2n+1)/24$

$$\frac{W_n^+/n^2 - E[W_n^+]/n^2}{\sqrt{Var(W_n^+)}/n^2} = \frac{\sqrt{n}\left(W_n^+/n^2 - E[W_n^+]/n^2\right)}{\sqrt{Var(W_n^+)/n^3}} = \frac{\sqrt{n}\,(T_n - \mu(\theta))}{\sigma(\theta)}$$

$\therefore E[W_n^+]/n^2 = \mu(\theta) \Leftrightarrow E[W_n^+] = n^2\mu(\theta)$

$Var(W_n^+)/n^3 = \sigma(\theta)^2 \Leftrightarrow Var(W_n^+) = n^3\sigma(\theta)^2$

☆ 〈Theorem 9.3.6 Approximation of Power and sample size in Signed rank test〉

Under the assumptions in 〈Theorem 9.3.4〉, the following approximation holds for the signed-rank test.

(a) (Approximation of power):

For the parameter under H_1, that is, $\theta_{1n} \simeq \theta_0 + K/\sqrt{n}$ $(K>0)$

$$\gamma_{\phi_{SR}}(\theta_{1n}) \simeq 1 - \Phi\left\{-\sqrt{n}\,(\theta_{1n}-\theta_0)\,\dot\mu(\theta_0)/\sigma(\theta_0) + z_\alpha\right\},$$

$$\dot\mu(\theta_0) = \int_{-\infty}^{+\infty} f^2(x)\,dx,\ \sigma^2(\theta_0) = 1/12$$ ☆ (practice deriving)

(b) (Approximation of sample size):

Sample size $N(W_n^+; \gamma, \theta_{1n})$ such that $\gamma_n(\theta_{1n}) \simeq \gamma$ is

$$N(W_n^+; \gamma, \theta_{1n}) \simeq \left(\sqrt{12}\int_{-\infty}^{+\infty} f^2(x)\,dx\right)^{-2} \cdot \left(\frac{z_\alpha - z_{1-\gamma}}{\theta_{1n}-\theta_0}\right)^2$$

(proof)

From 〈Theorem 9.3.5〉, $\mu(\theta)$ and $\sigma^2(\theta_0)$ are:

$$\mu(\theta) = \frac{1}{2}\int_{-\infty}^{+\infty} (1 - F(2\theta_0 - 2\theta - x))\,f(x)\,dx,$$

$$\sigma^2(\theta_0) = E[(1-F(-z))^2] - [E(1-F(-z))]^2,\ z\sim F$$
$$= E(U^2) - E(U)^2,\ U \sim U(0,1)$$

Thus, from 〈Theorem 9.3.1〉 (a) and (b) holds.

(My proof)

- Assumptions
1. X_1 and X_2 are iid random variables
2. Distribution of X is determined by a location parameter θ.
3. $f(x)$ and $F(x)$ are the PDF and CDF of the "standard" distribution (centered at 0, or where $\theta = 0$.)
4. Therefore, we can express X_i as $X_i = Z_i + \theta$, where $Z_i \sim f(z)$.

(a)

$P_\theta(X_1+X_2 > 2\theta_0)$

$= P_\theta(|Z_1+\theta| + |Z_2+\theta| > 2\theta_0)$

$= P_\theta(Z_1 + Z_2 > 2\theta_0 - 2\theta)$

Let $K = 2\theta_0 - 2\theta$

$= \int_{-\infty}^{+\infty} P_\theta(Z_1+Z_2 > K \mid Z_1 = x)\,f(x)\,dx$

$= \int_{-\infty}^{+\infty} P_\theta(x + Z_2 > K)\,f(x)\,dx$

$= \int_{-\infty}^{+\infty} P_\theta(Z_2 > K - x)\,f(x)\,dx$

$= \int_{-\infty}^{+\infty} [1 - F(K-x)]\,f(x)\,dx$

$= \int_{-\infty}^{+\infty} [1 - F(2\theta_0 - 2\theta - x)]\,f(x)\,dx$

$\therefore \mu(\theta) = \frac{1}{2}P_\theta(X_1+X_2 > 2\theta_0) = \frac{1}{2}\int_{-\infty}^{+\infty} [1 - F(2\theta_0 - 2\theta - x)]\,f(x)\,dx$

$\therefore \dot\mu(\theta) = \int_{-\infty}^{+\infty} f(2\theta_0 - 2\theta - x)\,f(x)\,dx$, $\dot\mu(\theta_0) = \int_{-\infty}^{+\infty} f(-x)f(x)\,dx$

$= \int_{-\infty}^{+\infty} f(x)^2\,dx$ $\because f(-x) = f(x)$

(b)

Let $A = I(Z_1+Z_2>0)$, $B = I(Z_1+Z_3>0)$

$\therefore \sigma^2(\theta_0) = Cov(A,B) = \underbrace{E[AB]}_{①} - \underbrace{E[A]E[B]}_{②}$

① $E[AB] = P(Z_1+Z_2>0,\ Z_1+Z_3>0)$

$E[AB] = E[E[AB \mid Z_1 = z]]$

$= \int_{-\infty}^{+\infty} P(Z_1+Z_2>0,\ Z_1+Z_3>0 \mid Z_1 = z)\,f(z)\,dz$

$= \int_{-\infty}^{+\infty} P(z+Z_2>0,\ z+Z_3>0)\,f(z)\,dz$

$= \int_{-\infty}^{+\infty} P(z+Z_2>0)\,P(z+Z_3>0)\,f(z)\,dz$

$= \int_{-\infty}^{+\infty} P(Z_2 > -z)\,P(Z_3 > -z)\,f(z)\,dz$

$= \int_{-\infty}^{+\infty} [1 - F(-z)]^2 f(z)\,dz$

$= E[(1-F(-z))^2]$

② $E[A] = E[E[A \mid Z_1 = z]]$

$= \int_{-\infty}^{+\infty} P(Z_1+Z_3>0 \mid Z_1 = z)\,f(z)\,dz$

$= \int_{-\infty}^{+\infty} P(z+Z_3>0)f(z)\,dz = \int_{-\infty}^{+\infty} P(Z_3 > -z)\,f(z)\,dz$

$= \int_{-\infty}^{+\infty} [1 - F(-z)]\,f(z)\,dz = E[1-F(-z)]$

$\therefore E[AB] - E[A]E[B] = E[(1-F(-z))^2] - E[1-F(-z)]^2$

The Signed-rank test assumes the underlying distribution of Z is symmetric about 0.

$\therefore F(x) = 1 - F(-x)$

$\therefore$ $E[AB] - E[A]E[B]$

$= E[\,[1-F(-z)]^2\,] - E[\,1-F(-z)\,]^2$

$= E[\,F(z)^2\,] - E[\,F(z)\,]^2$

$\overset{d}{=} E[\,U^2\,] - E[\,U\,]^2$, $U \sim U(0,1)$

$= \mathrm{Var}(U) = 1/12$

Ex. 9.3.5 ARE of Signed-rank test w.r.t. t-test

Suppose that the population distribution is continuous, symmetric about μ and its pdf is:

$$\frac{1}{\sigma} f\left(\frac{x-\mu}{\sigma}\right), \quad -\infty < \mu < +\infty, \ \sigma > 0$$

Consider testing the hypothesis:

$$H_0(\mu_0): \mu = \mu_0 \quad \text{vs} \quad H_1: \mu > \mu_0$$

at significance level $\alpha\ (0<\alpha<1)$ using the statistic,

$$W_n = \sum_{i=1}^{n} \mathrm{sgn}(X_i - \theta_0)\, R(|X_i - \theta_0|)$$

We can apply ⟨Theorem 9.3.4⟩.

"통계량 W_n을 이용하는 부호순위검정을 ⟨정리 9.3.4⟩와 같이 적용할 수 있다".

In this case, when $\mu_{1n} \simeq \mu_0 + K/\sqrt{n}$ $(K>0)$, the sample size needed for the power of the signed-rank test to be γ is, similar to ⟨Theorem 9.3.6⟩,

$$N(W_n; \gamma, \mu_{1n}) \simeq \left\{ \sqrt{12} \int_{-\infty}^{+\infty} f^2(z)\,dz / \sigma \right\}^{-2} \left(\frac{z_\alpha + z_{1-\gamma}}{\mu_{1n} - \mu_0} \right)^2$$

Meanwhile, from ⟨Ex.9.3.3⟩, the sample size needed in t-test is:

$$N(T_n; \gamma, \mu_{1n}) = \{1/\sqrt{\mathrm{Var}(X_1)}\}^{-2} \left(\frac{z_\alpha + z_{1-\gamma}}{\mu_{1n} - \mu_0} \right)^2,$$

$$\sqrt{\mathrm{Var}(X_1)} = \sigma \left(\int_{-\infty}^{+\infty} z^2 f(z)\,dz \right)^{1/2}$$

Hence the ARE of the signed-rank test w.r.t the t-test is given as:

$$\mathrm{ARE}(\{W_n\}, \{T_n\}) = 12 \left(\int_{-\infty}^{+\infty} f^2(z)\,dz \right)^2 \int_{-\infty}^{+\infty} z^2 f(z)\,dz$$

The values of the ARE under various population distributions are given in ⟨Table 9.3.2⟩. Comparing this table with ⟨Table 9.3.1⟩ for the sign test, we can see that the relative efficiency of the signed-rank test is much higher than that of the sign test. Even in the case of the normal distribution, its ARE reaches as high as 95%. That is, the signed-rank test is a very useful testing procedure, as it combines broad applicability – since it does not require assumptions about a specific form of the population distribution – with relatively high efficiency.

Table 9.3.2 ARE of signed-rank test w.r.t. t-test.

population Distribution	$N(\mu,\sigma^2)$	$L(\mu,\sigma)$	$DE(\mu,\sigma)$
$\mathrm{ARE}(\{S_n\}, \{T_n\})$	$3/\pi = 0.954\ldots$	$\pi^2/9 = 1.096$	1.5